

STEP Topic – Analysis

STEP I 1991 Question 9 (Pure)

- 9 (i) Suppose that the real number x satisfies the n inequalities

$$1 < x < 2$$

$$2 < x^2 < 3$$

$$3 < x^3 < 4$$

$$\vdots$$

$$n < x^n < n + 1$$

Prove without the use of a calculator that $n \leq 4$.

- (ii) If n is an integer strictly greater than 1, by considering how many terms there are in

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2},$$

or otherwise, show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1.$$

Hence or otherwise find, with justification, an integer N such that $\sum_{n=1}^N \frac{1}{n} > 10$.

STEP II 1987 Question 7 (Pure)

7 Prove that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^t \frac{x^{2n+2}}{1+x^2} dx.$$

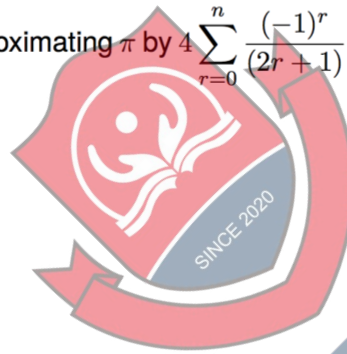
Hence show that, if $0 \leq t \leq 1$, then

$$\frac{t^{2n+3}}{2(2n+3)} \leq \left| \tan^{-1} t - \sum_{r=0}^n \frac{(-1)^r t^{2r+1}}{2r+1} \right| \leq \frac{t^{2n+3}}{2n+3}.$$

Show that, as $n \rightarrow \infty$,

$$4 \sum_{r=0}^n \frac{(-1)^r}{(2r+1)} \rightarrow \pi,$$

but that the error in approximating π by $4 \sum_{r=0}^n \frac{(-1)^r}{(2r+1)}$ is at least 10^{-2} if n is less than or equal to 98.



4Uadmission

STEP II Specimen Question 5 (Pure)

- 5 Explain, by means of a sketch, or otherwise, why

$$\sum_{r=n}^{\infty} \frac{1}{r^2} > \int_n^{\infty} \frac{dx}{x^2} > \sum_{r=n+1}^{\infty} \frac{1}{r^2}.$$

Deduce that

$$\frac{1}{n} > A - \sum_{r=1}^n \frac{1}{r^2} > \frac{1}{n+1}, \quad \text{where } A = \sum_{r=1}^{\infty} \frac{1}{r^2}.$$

Find the smallest value of n for which $\sum_{r=1}^n \frac{1}{r^2}$ approximates A with an error of less than 10^{-4} .

Show that, for this n ,

$$\frac{1}{n+1} + \sum_{r=1}^n \frac{1}{r^2}$$

approximates A with an error of less than 10^{-8} .



STEP II 1995 Question 4 (Pure)

4 Let

$$u_n = \int_0^{\frac{1}{2}\pi} \sin^n t \, dt$$

for each integer $n \geq 0$. By integrating

$$\int_0^{\frac{1}{2}\pi} \sin t \sin^{n-1} t \, dt$$

by parts, or otherwise, obtain a formula connecting u_n and u_{n-2} when $n \geq 2$ and deduce that

$$nu_n u_{n-1} = (n-1) u_{n-1} u_{n-2}$$

for all $n \geq 2$. Deduce that

$$nu_n u_{n-1} = \frac{1}{2}\pi.$$

Sketch graphs of $\sin^n t$ and $\sin^{n-1} t$, for $0 \leq t \leq \frac{1}{2}\pi$, on the same diagram and explain why $0 < u_n < u_{n-1}$. By using the result of the previous paragraph show that

$$nu_n^2 < \frac{1}{2}\pi < nu_{n-1}^2$$

for all $n \geq 1$. Hence show that

$$\left(\frac{n}{n+1}\right)^{\frac{1}{2}\pi} < nu_n^2 < \frac{1}{2}\pi$$

and deduce that $nu_n^2 \rightarrow \frac{1}{2}\pi$ as $n \rightarrow \infty$.

STEP I 1999 Question 8 (Pure)

- 8** The function f satisfies $0 \leq f(t) \leq K$ when $0 \leq t \leq x$. Explain by means of a sketch, or otherwise, why

$$0 \leq \int_0^x f(t) dt \leq Kx.$$

By considering $\int_0^1 \frac{t}{n(n-t)} dt$, or otherwise, show that, if $n > 1$,

$$0 \leq \ln \left(\frac{n}{n-1} \right) - \frac{1}{n} \leq \frac{1}{n-1} - \frac{1}{n}$$

and deduce that

$$0 \leq \ln N - \sum_{n=2}^N \frac{1}{n} \leq 1.$$

Deduce that as $N \rightarrow \infty$

$$\sum_{n=1}^N \frac{1}{n} \rightarrow \infty.$$

Noting that $2^{10} = 1024$, show also that if $N < 10^{30}$ then

$$\sum_{n=1}^N \frac{1}{n} < 101.$$

STEP II 2003 Question 7 (Pure)

7 Show that, if $n > 0$, then

$$\int_{e^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}} dx = \frac{2}{n^2 e}.$$

You may assume that $\frac{\ln x}{x} \rightarrow 0$ as $x \rightarrow \infty$.

Explain why, if $1 < a < b$, then

$$\int_b^{\infty} \frac{\ln x}{x^{n+1}} dx < \int_a^{\infty} \frac{\ln x}{x^{n+1}} dx.$$

Deduce that

$$\sum_{n=1}^N \frac{1}{n^2} < \frac{e}{2} \int_{e^{1/N}}^{\infty} \left(\frac{1 - x^{-N}}{x^2 - x} \right) \ln x dx,$$

where N is any integer greater than 1.



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STEP II 2012 Question 4 (Pure)

- 4** In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

- (i) Let n be an integer greater than 1. Show that, for any positive integer k ,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}.$$

Hence show that $\ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$. Deduce that

$$\left(1 + \frac{1}{n}\right)^n < e.$$

- (ii) Show, using an expansion in powers of $\frac{1}{y}$, that $\ln\left(\frac{2y+1}{2y-1}\right) > \frac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n ,

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}.$$

- (iii) Use parts (i) and (ii) to show that as $n \rightarrow \infty$

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e.$$

STEP III 2000 Question 7 (Pure)

7 Given that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots ,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < e$$

for any positive integer n .

The product $P(n)$ is defined, for any positive integer n , by

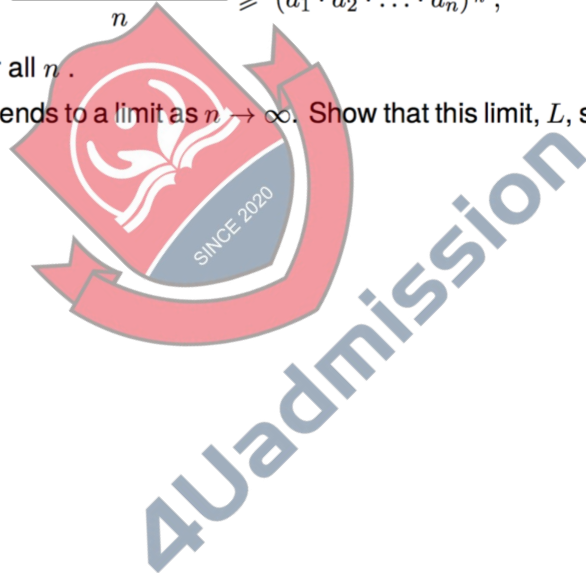
$$P(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \cdots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}},$$

to show that $P(n) < e$ for all n .

Explain briefly why $P(n)$ tends to a limit as $n \rightarrow \infty$. Show that this limit, L , satisfies $2 < L \leq e$.



STEP I 1995 Question 5 (Pure)

5 If

$$f(x) = nx - \binom{n}{2} \frac{x^2}{2} + \binom{n}{3} \frac{x^3}{3} - \cdots + (-1)^{r+1} \binom{n}{r} \frac{x^r}{r} + \cdots + (-1)^{n+1} \frac{x^n}{n},$$

show that

$$f'(x) = \frac{1 - (1-x)^n}{x}.$$

Deduce that

$$f(x) = \int_{1-x}^1 \frac{1-y^n}{1-y} dy.$$

Hence show that

$$f(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$



4Uadmission

STEP II 2006 Question 2 (Pure)

2 Using the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

show that $e > \frac{8}{3}$.

Show that $n! > 2^n$ for $n \geq 4$ and hence show that $e < \frac{67}{24}$.

Show that the curve with equation

$$y = 3e^{2x} + 14 \ln\left(\frac{4}{3} - x\right), \quad x < \frac{4}{3}$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and give a sketch to show the shape of the curve.



STEP I 1997 Question 6 (Pure)

- 6 Find constants $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and b such that

$$x^4(1-x)^4 = (a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(x^2 + 1) + b.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

Evaluate $\int_0^1 x^4(1-x)^4 dx$ and deduce that

$$\frac{22}{7} > \pi > \frac{22}{7} - \frac{1}{630}.$$



4Uadmission

STEP III 2004 Question 3 (Pure)

- 3** Given that $f''(x) > 0$ when $a \leq x \leq b$, explain with the aid of a sketch why

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x) \, dx < (b-a) \frac{f(a)+f(b)}{2}.$$

By choosing suitable a , b and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2} \left(\frac{1}{n^2} + \frac{1}{(n-1)^2} \right),$$

where n is an integer greater than 1.

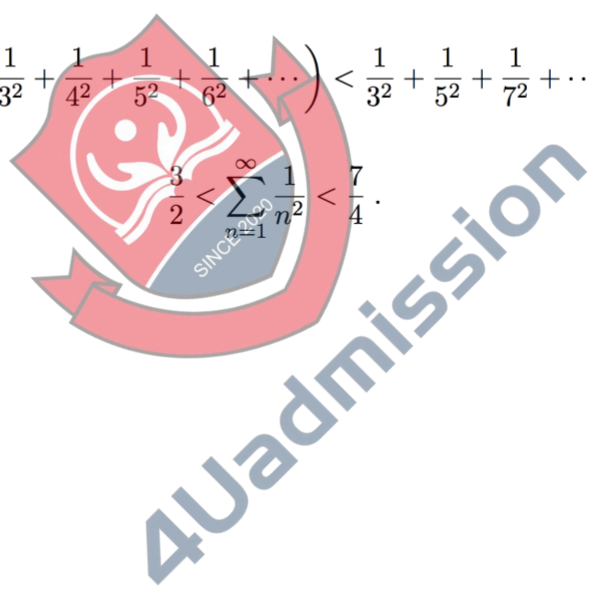
Deduce that

$$4 \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right).$$

Show that

$$\frac{1}{2} \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \dots \right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}.$$


STEP II 1999 Question 8 (Pure)

8 Prove that

$$\sum_{k=0}^n \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} . \quad (*)$$

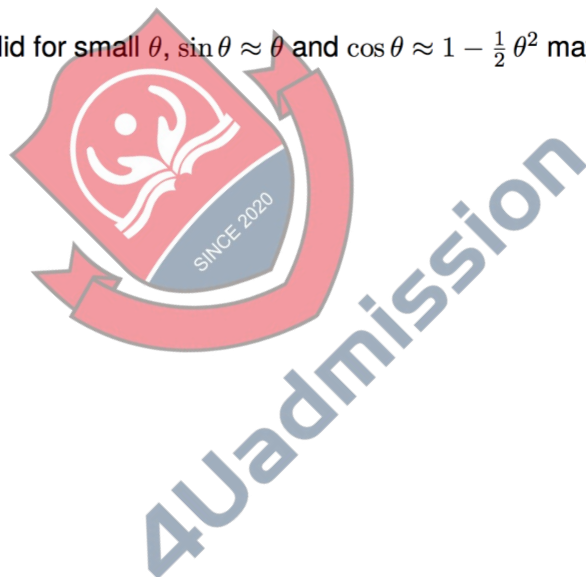
(i) Deduce that, when n is large,

$$\sum_{k=0}^n \sin \left(\frac{k\pi}{n} \right) \approx \frac{2n}{\pi} .$$

(ii) By differentiating (*) with respect to θ , or otherwise, show that, when n is large,

$$\sum_{k=0}^n k \sin^2 \left(\frac{k\pi}{2n} \right) \approx \left(\frac{1}{4} + \frac{1}{\pi^2} \right) n^2 .$$

[The approximations, valid for small θ , $\sin \theta \approx \theta$ and $\cos \theta \approx 1 - \frac{1}{2} \theta^2$ may be assumed.]



STEP I 1996 Question 6 (Pure)

6 Let $f(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}$ for $0 < x \leq \pi$.

(i) Using the formula

$$2 \sin \frac{1}{2}x \cos kx = \sin(k + \frac{1}{2})x - \sin(k - \frac{1}{2})x$$

(which you may assume), or otherwise, show that

$$f(x) = 1 + 2 \sum_{k=1}^n \cos kx.$$

(ii) Find $\int_0^\pi f(x) dx$ and $\int_0^\pi f(x) \cos x dx$.



4Uadmission

STEP III 1995 Question 4 (Pure)

4 Let

$$C_n(\theta) = \sum_{k=0}^n \cos k\theta$$

and let

$$S_n(\theta) = \sum_{k=0}^n \sin k\theta,$$

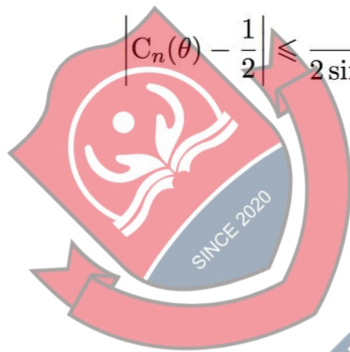
where n is a positive integer and $0 < \theta < 2\pi$. Show that

$$C_n(\theta) = \frac{\cos(\frac{1}{2}n\theta) \sin(\frac{1}{2}(n+1)\theta)}{\sin(\frac{1}{2}\theta)},$$

and obtain the corresponding expression for $S_n(\theta)$.

Hence, or otherwise, show that for $0 < \theta < 2\pi$,

$$\left| C_n(\theta) - \frac{1}{2} \right| \leq \frac{1}{2 \sin(\frac{1}{2}\theta)}.$$



4Uadmission

STEP II 1998 Question 3 (Pure)

- 3** Show that the sum S_N of the first N terms of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \cdots + \frac{2n-1}{n(n+1)(n+2)} + \cdots$$

is

$$\frac{1}{2} \left(\frac{3}{2} + \frac{1}{N+1} - \frac{5}{N+2} \right).$$

What is the limit of S_N as $N \rightarrow \infty$?

The numbers a_n are such that

$$\frac{a_n}{a_{n-1}} = \frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for a_n/a_1 and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1 = \frac{2}{9}$.



4Uadmission

STEP II 2015 Question 1 (Pure)

- 1 (i) By use of calculus, show that $x - \ln(1 + x)$ is positive for all positive x . Use this result to show that

$$\sum_{k=1}^n \frac{1}{k} > \ln(n + 1).$$

- (ii) By considering $x + \ln(1 - x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2.$$



STEP I 2011 Question 5 (Pure)

- 5 Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \quad (*)$$

has a unique solution in the range $0 < x < \pi$.

Let

$$I = \int_0^{\pi} |\sin x - kx| \, dx.$$

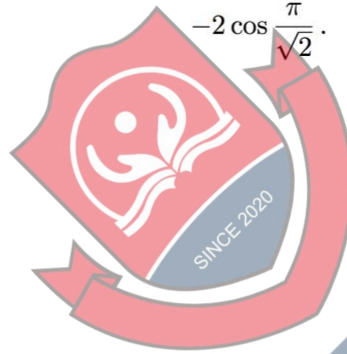
Show that

$$I = \frac{\pi^2 \sin \alpha}{2\alpha} - 2 \cos \alpha - \alpha \sin \alpha,$$

where α is the unique solution of (*).

Show that I , regarded as a function of α , has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

$$-2 \cos \frac{\pi}{\sqrt{2}}.$$



4Uadmission

STEP III 2014 Question 8 (Pure)

- 8 The numbers $f(r)$ satisfy $f(r) > f(r+1)$ for $r = 1, 2, \dots$. Show that, for any non-negative integer n ,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n)$$

where k is an integer greater than 1.

- (i) By taking $f(r) = 1/r$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

- (ii) By taking $f(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

- (iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n .

STEP III 2014 Question 6 (Pure)

6 Starting from the result that

$$h(t) > 0 \text{ for } 0 < t < x \implies \int_0^x h(t) dt > 0,$$

show that, if $f''(t) > 0$ for $0 < t < x_0$ and $f(0) = f'(0) = 0$, then $f(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}.$$



4Uadmission

STEP III 2015 Question 2 (Pure)

- 2** If s_1, s_2, s_3, \dots and t_1, t_2, t_3, \dots are sequences of positive numbers, we write

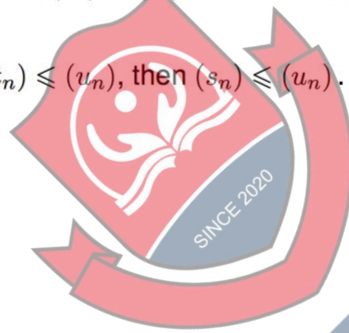
$$(s_n) \leq (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leq t_n$ whenever $n \geq m$ ".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m ; in the case of a false statement, you should give a counterexample.

- (i) $(1000n) \leq (n^2)$.
- (ii) If it is not the case that $(s_n) \leq (t_n)$, then it is the case that $(t_n) \leq (s_n)$.
- (iii) If $(s_n) \leq (t_n)$ and $(t_n) \leq (u_n)$, then $(s_n) \leq (u_n)$.
- (iv) $(n^2) \leq (2^n)$.



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