STEP Past Papers by Topic

STEP Topic – Analysis

STEP I 1991 Question 9 (Pure)

9 **(i)** Suppose that the real number x satisfies the n inequalities

$$\begin{aligned} 1 &< x &&< 2 \\ 2 &< x^2 &&< 3 \\ 3 &< x^3 &&< 4 \\ &\vdots \\ n &< x^n &&< n+1 \end{aligned}$$

Prove without the use of a calculator that $n \leqslant 4$.

(ii) If n is an integer strictly greater than 1, by considering how many terms there are in

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n^2},$$

or otherwise, show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1.$$

Hence or otherwise find, with justification, an integer N such that $\displaystyle\sum_{n=1}^{N} \frac{1}{n} > 10$.

STEP II 1987 Question 7 (Pure)

7 Prove that

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \cdots + \frac{(-1)^n t^{2n+1}}{2n+1} + (-1)^{n+1} \int_0^t \frac{x^{2n+2}}{1+x^2} \, \mathrm{d}x.$$

Hence show that, if $0 \leqslant t \leqslant 1$, then

$$\frac{t^{2n+3}}{2(2n+3)} \leqslant \left| \tan^{-1} t - \sum_{r=0}^{n} \frac{(-1)^r t^{2r+1}}{2r+1} \right| \leqslant \frac{t^{2n+3}}{2n+3}.$$

Show that, as $n \to \infty$,

$$4 \sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)} \to \pi,$$

but that the error in approximating π by $4 \sum_{r=0}^{n} \frac{(-1)^r}{(2r+1)}$ is at least 10^{-2} if n is less than or equal to 98.

STEP II Specimen Question 5 (Pure)

5 Explain, by means of a sketch, or otherwise, why

$$\sum_{r=n}^{\infty} \frac{1}{r^2} > \int_n^{\infty} \frac{\mathrm{d}x}{x^2} > \sum_{r=n+1}^{\infty} \frac{1}{r^2}.$$

Deduce that

$$\frac{1}{n} > A - \sum_{r=1}^{n} \frac{1}{r^2} > \frac{1}{n+1}, \qquad \text{where } A = \sum_{r=1}^{\infty} \frac{1}{r^2}.$$

Find the smallest value of n for which $\sum_{r=1}^{n} \frac{1}{r^2}$ approximates A with an error of less than 10^{-4}.

Show that, for this n,

$$\frac{1}{n+1} + \sum_{r=1}^{n} \frac{1}{r^2}$$

approximates A with an error of less than 10^{-8}.

STEP II 1995 Question 4 (Pure)

4 Let

$$u_n = \int_0^{\frac{1}{2}\pi} \sin^n t \, \mathrm{d}t$$

for each integer $n \geqslant 0$. By integrating

$$\int_0^{\frac{1}{2}\pi} \sin t \sin^{n-1} t \, \mathrm{d}t$$

by parts, or otherwise, obtain a formula connecting u_n and u_{n-2} when $n \geqslant 2$ and deduce that

$$n u_n u_{n-1} = (n-1)\, u_{n-1} u_{n-2}$$

for all $n \geqslant 2$. Deduce that

$$n u_n u_{n-1} = \tfrac{1}{2}\pi.$$

Sketch graphs of $\sin^n t$ and $\sin^{n-1} t$, for $0 \leqslant t \leqslant \frac{1}{2}\pi$, on the same diagram and explain why $0 < u_n < u_{n-1}$. By using the result of the previous paragraph show that

$$n u_n^2 < \tfrac{1}{2}\pi < n u_{n-1}^2$$

for all $n \geqslant 1$. Hence show that

$$\left(\frac{n}{n+1}\right)\tfrac{1}{2}\pi < n u_n^2 < \tfrac{1}{2}\pi$$

and deduce that $n u_n^2 \to \frac{1}{2}\pi$ as $n \to \infty$.

STEP I 1999 Question 8 (Pure)

8 The function f satisfies $0 \leqslant \mathrm{f}(t) \leqslant K$ when $0 \leqslant t \leqslant x$. Explain by means of a sketch, or otherwise, why

$$0 \leqslant \int_0^x \mathrm{f}(t)\,\mathrm{d}t \leqslant Kx.$$

By considering $\displaystyle\int_0^1 \frac{t}{n(n-t)}\,\mathrm{d}t$, or otherwise, show that, if $n > 1$,

$$0 \leqslant \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leqslant \frac{1}{n-1} - \frac{1}{n}$$

and deduce that

$$0 \leqslant \ln N - \sum_{n=2}^{N} \frac{1}{n} \leqslant 1.$$

Deduce that as $N \to \infty$

$$\sum_{n=1}^{N} \frac{1}{n} \to \infty.$$

Noting that $2^{10} = 1024$, show also that if $N < 10^{30}$ then

$$\sum_{n=1}^{N} \frac{1}{n} < 101.$$

STEP II 2003 Question 7 (Pure)

7 Show that, if $n > 0$, then

$$\int_{e^{1/n}}^{\infty} \frac{\ln x}{x^{n+1}}\, dx = \frac{2}{n^2 e}\,.$$

You may assume that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$.

Explain why, if $1 < a < b$, then

$$\int_{b}^{\infty} \frac{\ln x}{x^{n+1}}\, dx < \int_{a}^{\infty} \frac{\ln x}{x^{n+1}}\, dx\,.$$

Deduce that

$$\sum_{n=1}^{N} \frac{1}{n^2} < \frac{e}{2} \int_{e^{1/N}}^{\infty} \left(\frac{1 - x^{-N}}{x^2 - x} \right) \ln x \, dx\,,$$

where N is any integer greater than 1.

STEP II 2012 Question 4 (Pure)

4 In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}\,.$$

Hence show that $\ln\left(1+\dfrac{1}{n}\right) < \dfrac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}\,.$$

(ii) Show, using an expansion in powers of $\dfrac{1}{y}$, that $\ln\left(\dfrac{2y+1}{2y-1}\right) > \dfrac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}.$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}\,.$$

STEP III 2000 Question 7 (Pure)

7 Given that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots ,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < e$$

for any positive integer n.

The product $P(n)$ is defined, for any positive integer n, by

$$P(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \ldots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqslant (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} ,$$

to show that $P(n) < e$ for all n.

Explain briefly why $P(n)$ tends to a limit as $n \to \infty$. Show that this limit, L, satisfies $2 < L \leqslant e$.

STEP I 1995 Question 5 (Pure)

5 If

$$f(x) = nx - \binom{n}{2}\frac{x^2}{2} + \binom{n}{3}\frac{x^3}{3} - \cdots + (-1)^{r+1}\binom{n}{r}\frac{x^r}{r} + \cdots + (-1)^{n+1}\frac{x^n}{n},$$

show that

$$f'(x) = \frac{1-(1-x)^n}{x}.$$

Deduce that

$$f(x) = \int_{1-x}^{1} \frac{1-y^n}{1-y}\,dy.$$

Hence show that

$$f(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

STEP II 2006 Question 2 (Pure)

2 Using the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

show that $e > \frac{8}{3}$.

Show that $n! > 2^n$ for $n \geqslant 4$ and hence show that $e < \frac{67}{24}$.

Show that the curve with equation

$$y = 3e^{2x} + 14\ln(\tfrac{4}{3} - x)\,, \qquad x < \tfrac{4}{3}$$

has a minimum turning point between $x = \frac{1}{2}$ and $x = 1$ and give a sketch to show the shape of the curve.

STEP I 1997 Question 6 (Pure)

6 Find constants a_0, a_1, a_2, a_3, a_4, a_5, a_6 and b such that

$$x^4(1-x)^4 = (a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(x^2+1) + b.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x = \frac{22}{7} - \pi.$$

Evaluate $\displaystyle\int_0^1 x^4(1-x)^4\mathrm{d}x$ and deduce that

$$\frac{22}{7} > \pi > \frac{22}{7} - \frac{1}{630}.$$

STEP III 2004 Question 3 (Pure)

3 Given that $f''(x) > 0$ when $a \leqslant x \leqslant b$, explain with the aid of a sketch why

$$(b-a)\,f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\,\frac{f(a)+f(b)}{2}\,.$$

By choosing suitable a, b and $f(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1.

Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}\,.$$

STEP II 1999 Question 8 (Pure)

8 Prove that

$$\sum_{k=0}^{n} \sin k\theta = \frac{\cos \frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} \,. \qquad (*)$$

(i) Deduce that, when n is large,

$$\sum_{k=0}^{n} \sin\left(\frac{k\pi}{n}\right) \approx \frac{2n}{\pi} \,.$$

(ii) By differentiating $(*)$ with respect to θ, or otherwise, show that, when n is large,

$$\sum_{k=0}^{n} k\sin^2\left(\frac{k\pi}{2n}\right) \approx \left(\frac{1}{4}+\frac{1}{\pi^2}\right)n^2 \,.$$

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1-\frac{1}{2}\theta^2$ may be assumed.]

STEP I 1996 Question 6 (Pure)

6 Let $\mathrm{f}(x) = \dfrac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$ for $0 < x \leqslant \pi$.

(i) Using the formula

$$2\sin\tfrac{1}{2}x\cos kx = \sin(k+\tfrac{1}{2})x - \sin(k-\tfrac{1}{2})x$$

(which you may assume), or otherwise, show that

$$\mathrm{f}(x) = 1 + 2\sum_{k=1}^{n}\cos kx\,.$$

(ii) Find $\displaystyle\int_0^{\pi}\mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^{\pi}\mathrm{f}(x)\cos x\,\mathrm{d}x$.

STEP III 1995 Question 4 (Pure)

4 Let

$$\mathrm{C}_n(\theta) = \sum_{k=0}^{n} \cos k\theta$$

and let

$$\mathrm{S}_n(\theta) = \sum_{k=0}^{n} \sin k\theta,$$

where n is a positive integer and $0 < \theta < 2\pi$. Show that

$$\mathrm{C}_n(\theta) = \frac{\cos(\frac{1}{2}n\theta)\sin\left(\frac{1}{2}(n+1)\theta\right)}{\sin(\frac{1}{2}\theta)},$$

and obtain the corresponding expression for $\mathrm{S}_n(\theta)$.

Hence, or otherwise, show that for $0 < \theta < 2\pi$,

$$\left|\mathrm{C}_n(\theta) - \frac{1}{2}\right| \leqslant \frac{1}{2\sin(\frac{1}{2}\theta)}.$$

STEP II 1998 Question 3 (Pure)

3 Show that the sum S_N of the first N terms of the series

$$\frac{1}{1\cdot2\cdot3}+\frac{3}{2\cdot3\cdot4}+\frac{5}{3\cdot4\cdot5}+\cdots+\frac{2n-1}{n(n+1)(n+2)}+\cdots$$

is

$$\frac{1}{2}\left(\frac{3}{2}+\frac{1}{N+1}-\frac{5}{N+2}\right).$$

What is the limit of S_N as $N\to\infty$?

The numbers a_n are such that

$$\frac{a_n}{a_{n-1}}=\frac{(n-1)(2n-1)}{(n+2)(2n-3)}.$$

Find an expression for a_n/a_1 and hence, or otherwise, evaluate $\sum_{n=1}^{\infty} a_n$ when $a_1=\dfrac{2}{9}$.

STEP II 2015 Question 1 (Pure)

1 **(i)** By use of calculus, show that $x - \ln(1+x)$ is positive for all positive x. Use this result to show that

$$\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)\,.$$

(ii) By considering $x + \ln(1-x)$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 1 + \ln 2\,.$$

STEP I 2011 Question 5 (Pure)

5 Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

Let

$$I = \int_0^{\pi} \left| \sin x - kx \right| \mathrm{d}x\,.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

STEP III 2014 Question 8 (Pure)

8 The numbers $\mathrm{f}(r)$ satisfy $\mathrm{f}(r) > \mathrm{f}(r+1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer n,

$$k^n(k-1)\,\mathrm{f}(k^{n+1}) \leqslant \sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leqslant k^n(k-1)\,\mathrm{f}(k^n)$$

where k is an integer greater than 1.

(i) By taking $\mathrm{f}(r) = 1/r$, show that

$$\frac{N+1}{2} \leqslant \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leqslant N+1\,.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $\mathrm{f}(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leqslant 1\tfrac{1}{3}\,.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n.

STEP III 2014 Question 6 (Pure)

6 Starting from the result that

$$\mathrm{h}(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x \mathrm{h}(t)\,\mathrm{d}t > 0\,,$$

show that, if $\mathrm{f}''(t) > 0$ for $0 < t < x_0$ and $\mathrm{f}(0) = \mathrm{f}'(0) = 0$, then $\mathrm{f}(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

STEP III 2015 Question 2 (Pure)

2 If $s_1, s_2, s_3, \ldots$ and $t_1, t_2, t_3, \ldots$ are sequences of positive numbers, we write

$$(s_n) \leqslant (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leqslant t_n$ whenever $n \geqslant m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \leqslant (n^2)$.

(ii) If it is not the case that $(s_n) \leqslant (t_n)$, then it is the case that $(t_n) \leqslant (s_n)$.

(iii) If $(s_n) \leqslant (t_n)$ and $(t_n) \leqslant (u_n)$, then $(s_n) \leqslant (u_n)$.

(iv) $(n^2) \leqslant (2^n)$.