STEP Past Papers by Topic

STEP Topic – Binomial expansion

STEP I 1989 Question 5 (Pure)

5 Write down the binomial expansion of $(1+x)^n$, where n is a positive integer.

(i) By substituting particular values of x in the above expression, or otherwise, show that, if n is an even positive integer,

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots+\binom{n}{n-1}=2^{n-1}.$$

(ii) Show that, if n is any positive integer, then

$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n2^{n-1}.$$

Hence evaluate

$$\sum_{r=0}^{n}\left(r+(-1)^r\right)\binom{n}{r}.$$

STEP I 1988 Question 5 (Pure)

5 Given that $b > a > 0$, find, by using the binomial theorem, coefficients c_m $(m = 0, 1, 2, \ldots)$ such that

$$\frac{1}{(1-ax)(1-bx)} = c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m + \cdots$$

for $b\,|x| < 1$.

Show that

$$c_m^2 = \frac{a^{2m+2} - 2(ab)^{m+1} + b^{2m+2}}{(a-b)^2}.$$

Hence, or otherwise, show that

$$c_0^2 + c_1^2 x + c_2^2 x^2 + \cdots + c_m^2 x^m + \cdots = \frac{1+abx}{(1-abx)(1-a^2x)(1-b^2x)},$$

for x in a suitable interval which you should determine.

STEP II 1996 Question 1 (Pure)

1 **(i)** Find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4)^3.$$

You should set out your working clearly.

(ii) By considering the binomial expansions of $(1 + x)^{-2}$ and $(1 + x)^{-6}$, or otherwise, find the coefficient of x^6 in

$$(1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6)^3.$$

STEP II 1999 Question 4 (Pure)

4 By considering the expansions in powers of x of both sides of the identity

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n},$$

show that

$$\sum_{s=0}^{n} \binom{n}{s}^2 = \binom{2n}{n},$$

where $\displaystyle\binom{n}{s} = \frac{n!}{s!\,(n-s)!}$.

By considering similar identities, or otherwise, show also that:

(i) if n is an even integer, then

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s}^2 = (-1)^{n/2}\binom{n}{n/2};$$

(ii) $$\sum_{t=1}^{n} 2t\binom{n}{t}^2 = n\binom{2n}{n}.$$

STEP II 1999 Question 1 (Pure)

1 Let $x = 10^{100}$, $y = 10^x$, $z = 10^y$, and let

$$a_1 = x!, \quad a_2 = x^y, \quad a_3 = y^x, \quad a_4 = z^x, \quad a_5 = \mathrm{e}^{xyz}, \quad a_6 = z^{1/y}, \quad a_7 = y^{z/x}.$$

(i) Use Stirling's approximation $n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}}\mathrm{e}^{-n}$, which is valid for large n, to show that $\log_{10}(\log_{10} a_1) \approx 102$.

(ii) Arrange the seven numbers $a_1, \ldots, a_7$ in ascending order of magnitude, justifying your result.

STEP I 2003 Question 5 (Pure)

5 **(i)** In the binomial expansion of $(2x + 1/x^2)^6$ for $x \neq 0$, show that the term which is independent of x is 240.

Find the term which is independent of x in the binomial expansion of $(ax^3 + b/x^2)^{5n}$.

(ii) Let $\mathrm{f}(x) = (x^6 + 3x^5)^{1/2}$. By considering the expansion of $(1 + 3/x)^{1/2}$ show that the term which is independent of x in the expansion of $\mathrm{f}(x)$ in powers of $1/x$, for $|x| > 3$, is $27/16$.

Show that there is no term independent of x in the expansion of $\mathrm{f}(x)$ in powers of x, for $|x| < 3$.

STEP II 1988 Question 1 (Pure)

1 The function f is defined, for $x \neq 1$ and $x \neq 2$ by

$$\mathrm{f}(x) = \frac{1}{(x-1)(x-2)}$$

Show that for $|x| < 1$

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

and that for $1 < |x| < 2$

$$\mathrm{f}(x) = -\sum_{n=1}^{\infty} x^{-n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$

Find an expression for $\mathrm{f}(x)$ which is valid for $|x| > 2$.

STEP I 2011 Question 6 (Pure)

6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\frac{50}{128}+\cdots.$$

(ii) Find the sum of the series

STEP II 1998 Question 2 (Pure)

2 Use the first four terms of the binomial expansion of $(1 - 1/50)^{1/2}$, writing $1/50 = 2/100$ to simplify the calculation, to derive the approximation $\sqrt{2} \approx 1.414214$.

Calculate similarly an approximation to the cube root of 2 to six decimal places by considering $(1 + N/125)^{a}$, where a and N are suitable numbers.

[You need not justify the accuracy of your approximations.]

STEP II 1997 Question 12 (Probability and Statistics)

12 The game of Cambridge Whispers starts with the first participant Albert flipping an un-biased coin and whispering to his neighbour Bertha whether it fell 'heads' or 'tails'. Bertha then whispers this information to her neighbour, and so on. The game ends when the final player Zebedee whispers to Albert and the game is won, by all players, if what Albert hears is correct. The acoustics are such that the listeners have, independently at each stage, only a probability of 2/3 of hearing correctly what is said. Find the probability that the game is won when there are just three players.

By considering the binomial expansion of $(a+b)^n + (a-b)^n$, or otherwise, find a concise expression for the probability P that the game is won when is it played by n players each having a probability p of hearing correctly.

To avoid the trauma of a lost game, the rules are now modified to require Albert to whisper to Bertha what he hears from Zebedee, and so keep the game going, if what he hears from Zebedee is not correct. Find the expected total number of times that Albert whispers to Bertha before the modified game ends.

[You may use without proof the fact that $\sum_1^\infty kx^{k-1} = (1-x)^{-2}$ for $|x| < 1$.]

STEP II 2008 Question 2 (Pure)

2 Let a_n be the coefficient of x^n in the series expansion, in ascending powers of x, of

$$\frac{1+x}{(1-x)^2(1+x^2)},$$

where $|x| < 1$. Show, using partial fractions, that either $a_n = n+1$ or $a_n = n+2$ according to the value of n.

Hence find a decimal approximation, to nine significant figures, for the fraction $\dfrac{11\,000}{8181}$.
[You are not required to justify the accuracy of your approximation.]

STEP II 2014 Question 8 (Pure)

8 For positive integers n, a and b, the integer c_r $(0 \leqslant r \leqslant n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r, n, a and b.

For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \geqslant c_r$ for $0 \leqslant r \leqslant n$).

Show that

$$\frac{b(n+1)}{a+b} - 1 \leqslant m \leqslant \frac{b(n+1)}{a+b}\,.$$

Deduce that m is either a unique integer or one of two consecutive integers.

Let $\mathrm{G}(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

STEP II 2012 Question 1 (Pure)

1 Write down the general term in the expansion in powers of x of $(1-x^6)^{-2}$.

(i) Find the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}.$$

Obtain also, and simplify, formulae for the coefficient of x^n in the different cases that arise.

(ii) Show that the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}(1-x)^{-1}$$

is 55, and find the coefficients of x^{25} and x^{66}.