STEP Past Papers by Topic

STEP Topic – Circular Motion

STEP III Specimen Question 13 (Mechanics)

13 A particle of mass m is attached to a light circular hoop of radius a which is free to roll in a vertical plane on a rough horizontal table. Initially the hoop stands with the particle at its highest point and is then displaced slightly. Show that while the hoop is rolling on the table, the speed v of the particle when the radius to the particle makes an angle 2θ with the upward vertical is given by

$$v = 2(ga)^{\frac{1}{2}} \sin\theta.$$

Write down expressions in terms of θ for x, the horizontal displacement of the particle from its initial position, and y, its height above the table, and use them to show that

$$\theta = \tfrac{1}{2}(g/a)^{\frac{1}{2}} \tan\theta$$

and

$$\ddot{y} = -2g\sin^2\theta.$$

By considering the reaction of the table on the hoop, or otherwise, describe what happens to prevent the hoop rolling beyond the position for which $\theta = \pi/4$.

STEP I Specimen Question 13 (Mechanics)

13 A particle of mass m is attached to one end of a light elastic string of natural length a, and the other end of the string is attached to a fixed point A. When at rest and hanging vertically the string has length $2a$. The particle is set in motion so that it moves in a horizontal circle below the level of A. The vertical plane through A containing the string rotates with constant angular speed ω. Show that for this motion to be possible, the string must be stretched to a length greater than $2a$ and ω must satisfy

$$\frac{g}{2a} < \omega^2 < \frac{g}{a}.$$

STEP II Specimen Question 12 (Mechanics)

12 A thin smooth wire in the form of a circle, of radius a and centre O, is fixed in a horizontal plane. Two small beads A and B, each of mass m, are threaded on the wire and are connected by a light straight spring of natural length $2a\sin\alpha$ and modulus λ, where $0 < \alpha < \frac{1}{4}\pi$. The spring is compressed so that the angle AOB is 2β and the beads are then released from rest. Show that in the ensuing motion

$$ma\dot{\theta}^2\sin\alpha + \lambda(\sin\theta - \sin\alpha)^2 = \lambda(\sin\beta - \sin\alpha)^2$$

where 2θ denotes the angle AOB at time t after release.

(i) If $\beta - \alpha$ is small, show that T, the preiod of oscillations, is given approximately by

$$T = 2\pi\sqrt{\frac{ma\sin\alpha}{\lambda\cos^2\alpha}}.$$

(ii) If $\beta - \alpha$ is not small, write down an expression, in the form of a definite integral, for the exact period of oscillations, in each of the two cases (a) $\sin\beta > 2\sin\alpha - 1$ and (b) $\sin\beta < 2\sin\alpha - 1$.

STEP II 1990 Question 11 (Mechanics)

11 A disc is free to rotate in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2. Along one diameter is a narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is at a distance a from the axis and is moving towards the axis with speed V, where $k^2V^2 = \Omega^2a^2(k^2 + a^2)$. Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \qquad \text{and} \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\Omega r(k^2 + a^2)}{k(k^2 + r^2)^{\frac{1}{2}}}.$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}},$$

where θ is the angle through which the disc has turned at time t. By making the substitution $u = 1/r$, or otherwise, show that $r \sinh(\theta + \alpha) = k$, where $\sinh \alpha = k/a$. Hence, or otherwise, show that the particle never reaches the axis.

STEP III 1992 Question 12 (Mechanics)

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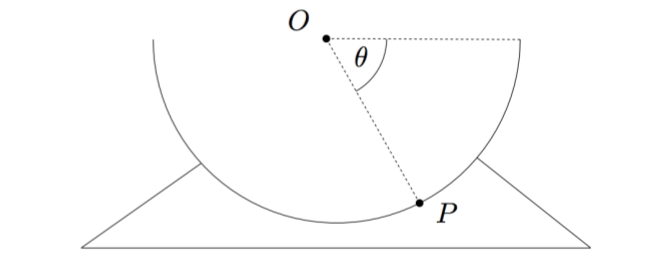


A smooth hemispherical bowl of mass $2m$ is rigidly mounted on a light carriage which slides freely on a horizontal table as shown in the diagram. The rim of the bowl is horizontal and has centre O. A particle P of mass m is free to slide on the inner surface of the bowl. Initially, P is in contact with the rim of the bowl and the system is at rest. The system is released and when OP makes an angle θ with the horizontal the velocity of the bowl is v? Show that

$$3v = a\dot{\theta}\sin\theta$$

and that

$$v^2 = \frac{2ga\sin^3\theta}{3(3 - \sin^2\theta)},$$

where a is the interior radius of the bowl.

Find, in terms of m, g and θ, the reaction between the bowl and the particle.

STEP II 1988 Question 11 (Mechanics)

11 A heavy particle lies on a smooth horizontal table, and is attached to one end of a light inextensible string of length L. The other end of the string is attached to a point P on the circumference of the base of a vertical post which is fixed into the table. The base of the post is a circle of radius a with its centre at a point O on the table. Initially, at time $t = 0$, the string is taut and perpendicular to the line OP. The particle is then struck in such a way that the string starts winding round the post and remains taut. At a later time t, a length $a\theta(t)$ $(< L)$ of the string is in contact with the post. Using cartesian axes with origin O, find the position and velocity vectors of the particle at time t in terms of a, L, θ and $\dot{\theta}$, and hence show that the speed of the particle is $(L - a\theta)\dot{\theta}$.

If the initial speed of the particle is v, show that the particle hits the post at a time $L^2/(2av)$.

STEP III 1995 Question 9 (Mechanics)

9 A thin circular disc of mass m, radius r and with its centre of mass at its centre C can rotate freely in a vertical plane about a fixed horizontal axis through a point O of its circumference. A particle P, also of mass m, is attached to the circumference of the disc so that the angle OCP is 2α, where $\alpha \leqslant \pi/2$.

(i) In the position of stable equilibrium OC makes an angle β with the vertical. Prove that

$$\tan\beta = \frac{\sin 2\alpha}{2 - \cos 2\alpha}.$$

(ii) The density of the disc at a point distant x from C is $\rho x/r$. Show that its moment of inertia about the horizontal axis through O is $8mr^2/5$.

(iii) The mid-point of CP is Q. The disc is held at rest with OQ horizontal and C lower than P and it is then released. Show that the speed v with which C is moving when P passes vertically below O is given by

$$v^2 = \frac{15gr\sin\alpha}{2(2+5\sin^2\alpha)}.$$

Find the maximum value of v^2 as α is varied.

STEP III 1999 Question 11 (Mechanics)

11 Calculate the moment of inertia of a uniform thin circular hoop of mass m and radius a about an axis perpendicular to the plane of the hoop through a point on its circumference.

The hoop, which is rough, rolls with speed v on a rough horizontal table straight towards the edge and rolls over the edge without initially losing contact with the edge. Show that the hoop will lose contact with the edge when it has rotated about the edge of the table through an angle θ, where

$$\cos\theta = \frac{1}{2} + \frac{v^2}{2ag}.$$

STEP III 2006 Question 10 (Mechanics)

10 A disc rotates freely in a horizontal plane about a vertical axis through its centre. The moment of inertia of the disc about this axis is mk^2 (where $k > 0$). Along one diameter is a smooth narrow groove in which a particle of mass m slides freely. At time $t = 0$, the disc is rotating with angular speed Ω, and the particle is a distance a from the axis and is moving with speed V along the groove, towards the axis, where $k^2V^2 = \Omega^2 a^2(k^2 + a^2)$.

Show that, at a later time t, while the particle is still moving towards the axis, the angular speed ω of the disc and the distance r of the particle from the axis are related by

$$\omega = \frac{\Omega(k^2 + a^2)}{k^2 + r^2} \quad \text{and} \quad \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\Omega^2 r^2 (k^2 + a^2)^2}{k^2(k^2 + r^2)} .$$

Deduce that

$$k\frac{\mathrm{d}r}{\mathrm{d}\theta} = -r(k^2 + r^2)^{\frac{1}{2}} ,$$

where θ is the angle through which the disc has turned by time t.

By making the substitution $u = k/r$, or otherwise, show that $r\sinh(\theta + \alpha) = k$, where $\sinh\alpha = k/a$. Deduce that the particle never reaches the axis.

STEP I 1994 Question 10 (Mechanics)

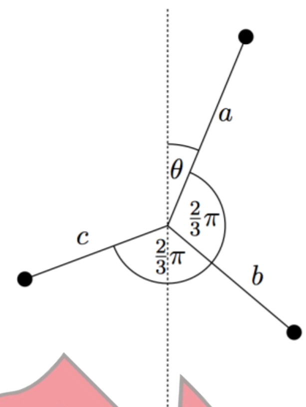
10 One end A of a light elastic string of natural length l and modulus of elasticity λ is fixed and a particle of mass m is attached to the other end B. The particle moves in a horizontal circle with centre on the vertical through A with angular velocity ω. If θ is the angle AB makes with the downward vertical, find an expression for $\cos\theta$ in terms of m, g, l, λ and ω.

Show that the motion described is possible only if

$$\frac{g\lambda}{l(\lambda + mg)} < \omega^2 < \frac{\lambda}{ml}.$$

STEP III 2005 Question 11 (Mechanics)

11 A horizontal spindle rotates freely in a fixed bearing. Three light rods are each attached by one end to the spindle so that they rotate in a vertical plane. A particle of mass m is fixed to the other end of each of the three rods. The rods have lengths a, b and c, with $a > b > c$ and the angle between any pair of rods is $\frac{2}{3}\pi$. The angle between the rod of length a and the vertical is θ, as shown in the diagram.



Find an expression for the energy of the system and show that, if the system is in equilibrium, then

$$\tan\theta = -\frac{(b-c)\sqrt{3}}{2a-b-c}\,.$$

Deduce that there are exactly two equilibrium positions and determine which of the two equilibrium positions is stable.

Show that, for the system to make complete revolutions, it must pass through its position of stable equilibrium with an angular velocity of at least

$$\sqrt{\frac{4gR}{a^2+b^2+c^2}}\,,$$

where $2R^2 = (a-b)^2 + (b-c)^2 + (c-a)^2$.

STEP III 2012 Question 10 (Mechanics)

10 A small ring of mass m is free to slide without friction on a hoop of radius a. The hoop is fixed in a vertical plane. The ring is connected by a light elastic string of natural length a to the highest point of the hoop. The ring is initially at rest at the lowest point of the hoop and is then slightly displaced. In the subsequent motion the angle of the string to the downward vertical is ϕ. Given that the ring first comes to rest just as the string becomes slack, find an expression for the modulus of elasticity of the string in terms of m and g.

Show that, throughout the motion, the magnitude R of the reaction between the ring and the hoop is given by

$$R = (12\cos^2\phi - 15\cos\phi + 5)mg$$

and that R is non-zero throughout the motion.

STEP II 1996 Question 10 (Pure)

10 The plot of 'Rhode Island Red and the Henhouse of Doom' calls for the heroine to cling on to the circumference of a fairground wheel of radius a rotating with constant angular velocity ω about its horizontal axis and then let go. Let ω_0 be the largest value of ω for which it is not possible for her subsequent path to carry her higher than the top of the wheel. Find ω_0 in terms of a and g.

If $\omega > \omega_0$ show that the greatest height above the top of the wheel to which she can rise is

$$\frac{a}{2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2.$$

STEP I 1997 Question 10 (Mechanics)

10 The point A is vertically above the point B. A light inextensible string, with a smooth ring P of mass m threaded onto it, has its ends attached at A and B. The plane APB rotates about AB with constant angular velocity ω so that P describes a horizontal circle of radius r and the string is taut. The angle BAP has value θ and the angle ABP has value ϕ. Show that

$$\tan \frac{\phi - \theta}{2} = \frac{g}{r\omega^2}.$$

Find the tension in the string in terms of m, g, r, ω and $\sin \frac{1}{2}(\theta + \phi)$.

STEP III 1989 Question 12 (Mechanics)

12 A smooth horizontal plane rotates with constant angular velocity Ω about a fixed vertical axis through a fixed point O of the plane. The point A is fixed in the plane and $OA = a$. A particle P lies on the plane and is joined to A by a light rod of length $b(> a)$ freely pivoted at A. Initially OAP is a straight line and P is moving with speed $(a + 2\sqrt{ab})\Omega$ perpendicular to OP in the same sense as Ω. At time t, AP makes an angle ϕ with OA produced. Obtain an expression for the component of the acceleration of P perpendicular to AP in terms of $\dfrac{\mathrm{d}^2\phi}{\mathrm{d}t^2}, \phi, a, b$ and Ω.

Hence find $\dfrac{\mathrm{d}\phi}{\mathrm{d}t}$, in terms of ϕ, a, b and Ω, and show that P never crosses OA.

STEP III 1997 Question 10 (Mechanics)

10 By pressing a finger down on it, a uniform spherical marble of radius a is made to slide along a horizontal table top with an initial linear velocity v_0 and an initial *backward* angular velocity ω_0 about the horizontal axis perpendicular to v_0. The frictional force between the marble and the table is constant (independent of speed).

For what value of $v_0/(a\omega_0)$ does the marble

(i) slide to a complete stop,

(ii) come to a stop and then roll back towards its initial position with linear speed $v_0/7$.

STEP III 2011 Question 11 (Mechanics)

11 A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin \phi = 2a \sin \tfrac{1}{2}\theta \, .$$

Show further that the magnitude of the couple is

$$\frac{mga^2 \sin \theta}{\sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}} \, .$$

The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta} \, .$$

STEP III 2004 Question 9 (Mechanics)

9 A circular hoop of radius a is free to rotate about a fixed horizontal axis passing through a point P on its circumference. The plane of the hoop is perpendicular to this axis. The hoop hangs in equilibrium with its centre, O, vertically below P. The point A on the hoop is vertically below O, so that POA is a diameter of the hoop.

A mouse M runs at constant speed u round the rough inner surface of the lower part of the hoop. Show that the mouse can choose its speed so that the hoop remains in equilibrium with diameter POA vertical.

Describe what happens to the hoop when the mouse passes the point at which angle $AOM = 2\arctan\mu$, where μ is the coefficient of friction between mouse and hoop.

STEP II 2014 Question 11 (Mechanics)

11 A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O, of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion.

Let x be the distance between O and the ring. Taking the y-axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x, L and α.

(i) By considering the vertical component of the equation of motion of P, show that

$$km\ddot{x}\cos\alpha = T\cos\alpha - kmg\,,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R.

(ii) Show that $\dfrac{\sin\alpha}{(1-\sin\alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k.

(iii) Show that $\ddot{x} = -g\tan\alpha\,.$

STEP III 2013 Question 10 (Mechanics)

10 A uniform rod AB has mass M and length $2a$. The point P lies on the rod a distance $a - x$ from A. Show that the moment of inertia of the rod about an axis through P and perpendicular to the rod is

$$\tfrac{1}{3}M(a^2 + 3x^2)\,.$$

The rod is free to rotate, in a horizontal plane, about a fixed vertical axis through P. Initially the rod is at rest. The end B is struck by a particle of mass m moving horizontally with speed u in a direction perpendicular to the rod. The coefficient of restitution between the rod and the particle is e. Show that the angular velocity of the rod immediately after impact is

$$\frac{3mu(1+e)(a+x)}{M(a^2+3x^2)+3m(a+x)^2}\,.$$

In the case $m = 2M$, find the value of x for which the angular velocity is greatest and show that this angular velocity is $u(1+e)/a$.

STEP III 2008 Question 11 (Mechanics)

11 A circular wheel of radius r has moment of inertia I about its axle, which is fixed in a horizontal position. A light string is wrapped around the circumference of the wheel and a particle of mass m hangs from the free end. The system is released from rest and the particle descends. The string does not slip on the wheel.

As the particle descends, the wheel turns through n_1 revolutions, and the string then detaches from the wheel. At this moment, the angular speed of the wheel is ω_0. The wheel then turns through a further n_2 revolutions, in time T, before coming to rest. The couple on the wheel due to resistance is constant.

Show that

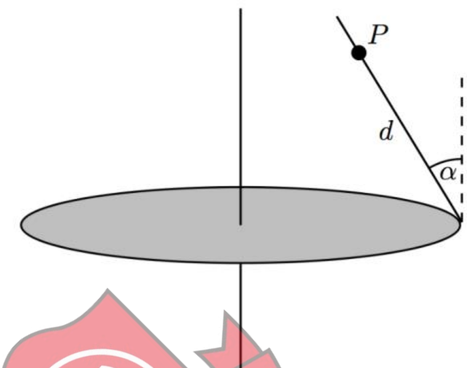
$$\frac{1}{2}\omega_0 T = 2\pi n_2$$

and

$$I = \frac{mgrn_1T^2 - 4\pi mr^2n_2^2}{4\pi n_2(n_1 + n_2)}\ .$$

STEP III 20151 Question 11 (Mechanics)

11 **(i)** A horizontal disc of radius r rotates about a vertical axis through its centre with angular speed ω. One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle P of mass m is attached to the rod at a distance d from the hinge. The rod makes a constant angle α with the upward vertical, as shown in the diagram, and $d\sin\alpha < r$.



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by P is parallel to the rod.

Show also that

$$r\cot\alpha = a + d\cos\alpha\,,$$

where $a = \dfrac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of m, g and α.

(ii) The disc and rod rotate as in part (i), but two particles (instead of P) are attached to the rod. The masses of the particles are m_1 and m_2 and they are attached to the rod at distances d_1 and d_2 from the hinge, respectively. The rod makes a constant angle β with the upward vertical and $d_1\sin\beta < d_2\sin\beta < r$. Show that β satisfies an equation of the form

$$r\cot\beta = a + b\cos\beta\,,$$

where b should be expressed in terms of d_1, d_2, m_1 and m_2.

STEP III 20141 Question 11 (Mechanics)

11 A particle P of mass m is connected by two light inextensible strings to two fixed points A and B, with A vertically above B. The string AP has length x. The particle is rotating about the vertical through A and B with angular velocity ω, and both strings are taut. Angles PAB and PBA are α and β, respectively.

Find the tensions T_A and T_B in the strings AP and BP (respectively), and hence show that $\omega^2 x \cos\alpha \geqslant g$.

Consider now the case that $\omega^2 x \cos\alpha = g$. Given that $AB = h$ and $BP = d$, where $h > d$, show that $h\cos\alpha \geqslant \sqrt{h^2 - d^2}$. Show further that

$$mg < T_A \leqslant \frac{mgh}{\sqrt{h^2 - d^2}}\,.$$

Describe the geometry of the strings when T_A attains its upper bound.
