STEP Past Papers by Topic

STEP Topic – Coin toss

STEP III 1997 Question 12 (Probability and Statistics)

12 **(i)** I toss a biased coin which has a probability p of landing heads and a probability $q = 1 - p$ of landing tails. Let K be the number of tosses required to obtain the first head and let

$$\mathrm{G}(s) = \sum_{k=1}^{\infty} \mathrm{P}(K = k)s^k.$$

Show that

$$\mathrm{G}(s) = \frac{ps}{1 - qs}$$

and hence find the expectation and variance of K.

(ii) I sample cards at random with replacement from a normal pack of 52. Let N be the total number of draws I make in order to sample every card at least once. By expressing N as a sum $N = N_1 + N_2 + \cdots + N_{52}$ of random variables, or otherwise, find the expectation of N. Estimate the numerical value of this expectation, using the approximations $\mathrm{e} \approx 2.7$ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx 0.5 + \ln n$ if n is large.

STEP I 2018 Question 12 (Probability and Statistics)

12 A bag contains three coins. The probabilities of their showing heads when tossed are p_1, p_2 and p_3.

(i) A coin is taken at random from the bag and tossed. What is the probability that it shows a head?

(ii) A coin is taken at random from the bag (containing three coins) and tossed; the coin is returned to the bag and again a coin is taken at random from the bag and tossed. Let N_1 be the random variable whose value is the number of heads shown on the two tosses. Find the expectation of N_1 in terms of p, where $p = \frac{1}{3}(p_1 + p_2 + p_3)$, and show that $\text{Var}(N_1) = 2p(1-p)$.

(iii) Two of the coins are taken at random from the bag (containing three coins) and tossed. Let N_2 be the random variable whose value is the number of heads showing on the two coins. Find $\text{E}(N_2)$ and $\text{Var}(N_2)$.

(iv) Show that $\text{Var}(N_2) \leqslant \text{Var}(N_1)$, with equality if and only if $p_1 = p_2 = p_3$.

STEP III 2001 Question 13 (Probability and Statistics)

13 In a game for two players, a fair coin is tossed repeatedly. Each player is assigned a sequence of heads and tails and the player whose sequence appears first wins. Four players, A, B, C and D take turns to play the game. Each time they play, A is assigned the sequence TTH (i.e. Tail then Tail then Head), B is assigned THH, C is assigned HHT and D is assigned HTT.

(i) A and B play the game. Let p_{HH}, p_{HT}, p_{TH} and p_{TT} be the probabilities of A winning the game given that the first two tosses of the coin show HH, HT, TH and TT, respectively. Explain why $p_{\text{TT}} = 1$, and why $p_{\text{HT}} = \frac{1}{2}p_{\text{TH}} + \frac{1}{2}p_{\text{TT}}$. Show that $p_{\text{HH}} = p_{\text{HT}} = \frac{2}{3}$ and that $p_{\text{TH}} = \frac{1}{3}$. Deduce that the probability that A wins the game is $\frac{2}{3}$.

(ii) B and C play the game. Find the probability that B wins.

(iii) Show that if C plays D, then C is more likely to win than D, but that if D plays A, then D is more likely to win than A.

STEP II 2015 Question 12 (Probability and Statistics)

12 Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT?

Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$.

Find the probability that C wins.

STEP II 2002 Question 12 (Probability and Statistics)

12 On K consecutive days each of L identical coins is thrown M times. For each coin, the probability of throwing a head in any one throw is p (where $0 < p < 1$). Show that the probability that on exactly k of these days more than l of the coins will each produce fewer than m heads can be approximated by

$$\binom{K}{k} q^k (1-q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h - 2l - 1}{2\sqrt{h}}\right), \qquad h = L\Phi\left(\frac{2m - 1 - 2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(.)$ is the cumulative distribution function of a standard normal variate.

Would you expect this approximation to be accurate in the case $K = 7$, $k = 2$, $L = 500$, $l = 4$,

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STEP I 2016 Question 12 (Probability and Statistics)

12 **(i)** Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.

(ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.

(iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n+1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2, and hence obtain a generalisation of the results of parts (i) and (ii).

STEP III 2002 Question 12 (Probability and Statistics)

12 In a game, a player tosses a biased coin repeatedly until two successive tails occur, when the game terminates. For each head which occurs the player wins £1. If E is the expected number of tosses of the coin in the course of a game, and p is the probability of a head, explain why

$$E = p\,(1+E) + (1-p)\,p\,(2+E) + 2\,(1-p)^2\ ,$$

and hence determine E in terms of p. Find also, in terms of p, the expected winnings in the course of a game.

A second game is played, with the same rules, except that the player continues to toss the coin until r successive tails occur. Show that the expected number of tosses in the course of a game is given by the expression $\dfrac{1-q^r}{pq^r}$, where $q = 1-p$.

STEP III 2009 Question 12 (Probability and Statistics)

12 **(i)** Albert tosses a fair coin k times, where k is a given positive integer. The number of heads he gets is X_1. He then tosses the coin X_1 times, getting X_2 heads. He then tosses the coin X_2 times, getting X_3 heads. The random variables X_4, X_5, ... are defined similarly. Write down $\mathrm{E}(X_1)$.

By considering $\mathrm{E}(X_2 \mid X_1 = x_1)$, or otherwise, show that $\mathrm{E}(X_2) = \frac{1}{4}k$.

Find $\displaystyle\sum_{i=1}^{\infty} \mathrm{E}(X_i)$.

(ii) Bertha has k fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is Y_1. She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is Y_2. The random variables $Y_3, Y_4, \ldots, Y_k$ are defined similarly, and $Y = \sum_{i=1}^{k} Y_i$.

Obtain the probability generating function of Y, and use it to find $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

STEP II 2013 Question 13 (Mechanics)

13 A biased coin has probability p of showing a head and probability q of showing a tail, where $p \neq 0$, $q \neq 0$ and $p \neq q$. When the coin is tossed repeatedly, runs occur. A *straight run* of length n is a sequence of n consecutive heads or n consecutive tails. An *alternating run* of length n is a sequence of length n alternating between heads and tails. An alternating run can start with either a head or a tail.

Let S be the length of the longest straight run beginning with the first toss and let A be the length of the longest alternating run beginning with the first toss.

(i) Explain why $\mathrm{P}(A = 1) = p^2 + q^2$ and find $\mathrm{P}(S = 1)$. Show that $\mathrm{P}(S = 1) < \mathrm{P}(A = 1)$.

(ii) Show that $\mathrm{P}(S = 2) = \mathrm{P}(A = 2)$ and determine the relationship between $\mathrm{P}(S = 3)$ and $\mathrm{P}(A = 3)$.

(iii) Show that, for $n > 1$, $\mathrm{P}(S = 2n) > \mathrm{P}(A = 2n)$ and determine the corresponding relationship between $\mathrm{P}(S = 2n + 1)$ and $\mathrm{P}(A = 2n + 1)$. [You are advised *not* to use $p + q = 1$ in this part.]

STEP I 1996 Question 14 (Probability and Statistics)

14 A biased coin, with a probability p of coming up heads and a probability $q = 1 - p$ of coming up tails, is tossed repeatedly. Let A be the event that the first run of r successive heads occurs before the first run of s successive tails. If H is the even that on the first toss the coin comes up heads and T is the event that it comes up tails, show that

$$\mathrm{P}(A|H) = p^{\alpha} + (1 - p^{\alpha})\mathrm{P}(A|T),$$
$$\mathrm{P}(A|T) = (1 - q^{\beta})\mathrm{P}(A|H),$$

where α and β are to be determined. Use these two equations to find $\mathrm{P}(A|H)$, $\mathrm{P}(A|T)$, and hence $\mathrm{P}(A)$.

STEP II 2001 Question 14 (Mechanics)

14 Two coins A and B are tossed together. A has probability p of showing a head, and B has probability $2p$, independent of A, of showing a head, where $0 < p < \frac{1}{2}$. The random variable X takes the value 1 if A shows a head and it takes the value 0 if A shows a tail. The random variable Y takes the value 1 if B shows a head and it takes the value 0 if B shows a tail. The random variable T is defined by

$$T = \lambda X + \tfrac{1}{2}(1-\lambda)Y.$$

Show that $\mathrm{E}(T) = p$ and find an expression for $\mathrm{Var}\,(T)$ in terms of p and λ. Show that as λ varies, the minimum of $\mathrm{Var}\,(T)$ occurs when

$$\lambda = \frac{1-2p}{3-4p}\,.$$

The two coins are tossed n times, where $n > 30$, and $\overline{T}$ is the mean value of T. Let b be a fixed positive number. Show that the maximum value of $\mathrm{P}\big(|\overline{T}-p| < b\big)$ as λ varies is approximately $2\Phi(b/s) - 1$, where Φ is the cumulative distribution function of a standard normal variate and

$$s^2 = \frac{p(1-p)(1-2p)}{(3-4p)n}\,.$$

STEP I 1990 Question 15 (Probability and Statistics)

15 A coin has probability p $(0 < p < 1)$ of showing a head when tossed. Give a careful argument to show that the kth head in a series of consecutive tosses is achieved after *exactly* n tosses with probability

$$\binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad (n \geqslant k).$$

Given that it took an even number of tosses to achieve exactly $k-1$ heads, find the probability that exactly k heads are achieved after an even number of tosses.

If this coin is tossed until exactly 3 heads are obtained, what is the probability that *exactly* 2 of the heads occur on even-numbered tosses?