STEP Past Papers by Topic

STEP Topic – Continuous random variable

STEP III 1994 Question 13 (Probability and Statistics)

13 During his performance a trapeze artist is supported by two identical ropes, either of which can bear his weight. Each rope is such that the time, in hours of performance, before it fails is exponentially distributed, independently of the other, with probability density function $\lambda\exp(-\lambda t)$ for $t \geqslant 0$ (and 0 for $t < 0$), for some $\lambda > 0$. A particular rope has already been in use for t_0 hours of performance. Find the distribution for the length of time the artist can continue to use it before it fails. Interpret and comment upon your result.

Before going on tour the artist insists that the management purchase two new ropes of the above type. Show that the probability density function of the time until both ropes fail is

$$\mathrm{f}(t) = \begin{cases} 2\lambda \mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t}) & \text{if } t \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

If each performance lasts for h hours, find the probability that both ropes fail during the nth performance. Show that the probability that both ropes fail during the same performance is $\tanh(\lambda h/2)$.

STEP III 2002 Question 13 (Probability and Statistics)

13 A continuous random variable is said to have an exponential distribution with parameter λ if its density function is $\mathrm{f}(t) = \lambda \mathrm{e}^{-\lambda t} \quad (0 \leqslant t < \infty)$. If X_1 and X_2, which are independent random variables, have exponential distributions with parameters λ_1 and λ_2 respectively, find an expression for the probability that either X_1 or X_2 (or both) is less than x. Prove that if X is the random variable whose value is the lesser of the values of X_1 and X_2, then X also has an exponential distribution.

Route A and Route B buses run from my house to my college. The time between buses on each route has an exponential distribution and the mean time between buses is 15 minutes for Route A and 30 minutes for Route B. The timings of the buses on the two routes are independent. If I emerge from my house one day to see a Route A bus and a Route B bus just leaving the stop, show that the median wait for the next bus to my college will be approximately 7 minutes.

STEP II 1998 Question 13 (Mechanics)

13 A random variable X has the probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geqslant 0, \\ 0 & x < 0. \end{cases}$$

Show that

$$P(X > s + t \,|\, X > t) = P(X > s).$$

The time it takes an assistant to serve a customer in a certain shop is a random variable with the above distribution and the times for different customers are independent. If, when I enter the shop, the only two assistants are serving one customer each, what is the probability that these customers are both still being served at time t after I arrive?

One of the assistants finishes serving his customer and immediately starts serving me. What is the probability that I am still being served when the other customer has finished being served?

STEP II 2009 Question 12 (Mechanics)

12 A continuous random variable X has probability density function given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x < 0 \\ k\mathrm{e}^{-2x^2} & \text{for } 0 \leqslant x < \infty \, , \end{cases}$$

where k is a constant.

(i) Sketch the graph of $\mathrm{f}(x)$.

(ii) Find the value of k.

(iii) Determine $\mathrm{E}(X)$ and $\mathrm{Var}\,(X)$.

(iv) Use statistical tables to find, to three significant figures, the median value of X.

STEP II 2010 Question 12 (Probability and Statistics)

12 The continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a-b)}\,.$$

(ii) Show that the median, M, of X is given by $M = \dfrac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

STEP III 1999 Question 13 (Probability and Statistics)

13 The cakes in our canteen each contain exactly four currants, each currant being randomly placed in the cake. I take a proportion X of a cake where X is a random variable with density function

$$f(x) = Ax$$

for $0 \leqslant x \leqslant 1$ where A is a constant.

(i) What is the expected number of currants in my portion?

(ii) If I find all four currants in my portion, what is the probability that I took more than half the cake?

STEP I 2011 Question 13 (Probability and Statistics)

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.

STEP I 1993 Question 14 (Probability and Statistics)

14 When he sets out on a drive Mr Toad selects a speed V kilometres per minute where V is a random variable with probability density

$$\alpha v^{-2} \mathrm{e}^{-\alpha v^{-1}}$$

and α is a strictly positive constant. He then drives at constant speed, regardless of other drivers, road conditions and the Highway Code. The traffic lights at the Wild Wood cross-roads change from red to green when Mr Toad is exactly 1 kilometre away in his journey towards them. If the traffic light is green for g minutes, then red for r minutes, then green for g minutes, and so on, show that the probability that he passes them after $n(g+r)$ minutes but before $n(g+r)+g$ minutes, where n is a positive integer, is

$$\mathrm{e}^{-\alpha n(g+r)} - \mathrm{e}^{-\alpha(n(g+r))+g}.$$

Find the probability $\mathrm{P}(\alpha)$ that he passes the traffic lights when they are green.

Show that $\mathrm{P}(\alpha) \to 1$ as $\alpha \to \infty$ and, by noting that $(\mathrm{e}^x - 1)/x \to 1$ as $x \to 0$, or otherwise, show that

$$\mathrm{P}(\alpha) \to \frac{g}{r+g} \quad \text{as } \alpha \to 0.$$

[NB: the traffic light show only green and red - not amber.]

STEP I 1989 Question 14 (Probability and Statistics)

14 The prevailing winds blow in a constant southerly direction from an enchanted castle. Each year, according to an ancient tradiction, a princess releases 96 magic seeds from the castle, which are carried south by the wind before falling to rest. South of the castle lies one league of grassy parkland, then one league of lake, then one league of farmland, and finally the sea. If a seed falls on land it will immediately grow into a fever tree. (Fever trees do not grow in water). Seeds are blown independently of each other. The random variable L is the distance in leagues south of the castle at which a seed falls to rest (either on land or water). It is known that the probability density function f of L is given by

$$\mathrm{f}(x) = \begin{cases} \frac{1}{2} - \frac{1}{8}x & \text{for } 0 \leqslant x \leqslant 4, \\ 0 & \text{otherwise.} \end{cases}$$

What is the mean number of fever trees which begin to grow each year?

(i) The random variable Y is defined as the distance in leagues south of the castle at which a new fever tree grows from a seed carried by the wind. Sketch the probability density function of Y, and find the mean of Y.

(ii) One year messengers bring the king the news that 23 new fever trees have grown in the farmland. The wind never varies, and so the king suspects that the ancient tradition have not been followed properly. Is he justified in his suspicions?

STEP II 2006 Question 14 (Mechanics)

14 Sketch the graph of $y = \dfrac{1}{x \ln x}$ for $x > 0,\ x \neq 1$. You may assume that $x \ln x \to 0$ as $x \to 0$.

The continuous random variable X has probability density function

$$\mathrm{f}(x) = \begin{cases} \dfrac{\lambda}{x \ln x} & \text{for } a \leqslant x \leqslant b\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where a, b and λ are suitably chosen constants.

(i) In the case $a = 1/4$ and $b = 1/2$, find λ.

(ii) In the case $\lambda = 1$ and $a > 1$, show that $b = a^{\mathrm{e}}$.

(iii) In the case $\lambda = 1$ and $a = \mathrm{e}$, show that $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2) \approx \frac{31}{108}$.

(iv) In the case $\lambda = 1$ and $a = \mathrm{e}^{1/2}$, find $\mathrm{P}(\mathrm{e}^{3/2} \leqslant X \leqslant \mathrm{e}^2)$.

STEP II 1996 Question 14 (Mechanics)

14 The random variable X is uniformly distributed on $[0, 1]$. A new random variable Y is defined by the rule

$$Y = \begin{cases} 1/4 & \text{if } X \leqslant 1/4, \\ X & \text{if } 1/4 \leqslant X \leqslant 3/4 \\ 3/4 & \text{if } X \geqslant 3/4. \end{cases}$$

Find $\mathrm{E}(Y^n)$ for all integers $n \geqslant 1$.

Show that $\mathrm{E}(Y) = \mathrm{E}(X)$ and that

$$\mathrm{E}(X^2) - \mathrm{E}(Y^2) = \frac{1}{24}.$$

By using the fact that $4^n = (3+1)^n$, or otherwise, show that $\mathrm{E}(X^n) > \mathrm{E}(Y^n)$ for $n \geqslant 2$.

Suppose that $Y_1, Y_2, \ldots$ are independent random variables each having the same distribution as Y. Find, to a good approximation, K such that

$$\mathrm{P}(Y_1 + Y_2 + \cdots + Y_{240000} < K) = 3/4.$$

STEP III 2005 Question 14 (Probability and Statistics)

14 In this question, you may use the result

$$\int_0^\infty \frac{t^m}{(t+k)^{n+2}}\,\mathrm{d}t = \frac{m!\,(n-m)!}{(n+1)!\,k^{n-m+1}}\,,$$

where m and n are positive integers with $n \geqslant m$, and where $k > 0$.

The random variable V has density function

$$\mathrm{f}(x) = \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}} \qquad (0 \leqslant x < \infty)\,,$$

where a is a positive integer. Show that $C = \dfrac{(2a+1)!}{a!\,a!}$.

Show, by means of a suitable substitution, that

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}}\,\mathrm{d}x = \int_{\frac{k^2}{v}}^\infty \frac{u^a}{(u+k)^{2a+2}}\,\mathrm{d}u$$

and deduce that the median value of V is k. Find the expected value of V.

The random variable V represents the speed of a randomly chosen gas molecule. The time taken for such a particle to travel a fixed distance s is given by the random variable $T = \dfrac{s}{V}$.

Show that

$$\mathrm{P}(T < t) = \int_{\frac{s}{t}}^\infty \frac{C\,k^{a+1}\,x^a}{(x+k)^{2a+2}}\,\mathrm{d}x \qquad (*)$$

and hence find the density function of T. You may find it helpful to make the substitution $u = \dfrac{s}{x}$ in the integral $(*)$.

Hence show that the product of the median time and the median speed is equal to the distance s, but that the product of the expected time and the expected speed is greater than s.

STEP I 1999 Question 14 (Probability and Statistics)

14 When I throw a dart at a target, the probability that it lands a distance X from the centre is a random variable with density function

$$f(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

I score points according to the position of the dart as follows: if $0 \leqslant X < \frac{1}{4}$, my score is 4; if $\frac{1}{4} \leqslant X < \frac{1}{2}$, my score is 3; if $\frac{1}{2} \leqslant X < \frac{3}{4}$, my score is 2; if $\frac{3}{4} \leqslant X \leqslant 1$, my score is 1.

(i) Show that my expected score from one dart is 15/8.

(ii) I play a game with the following rules. I start off with a total score 0, and each time I throw a dart my score on that throw is added to my total. Then:

if my new total is greater than 3, I have lost and the game ends;
if my new total is 3, I have won and the game ends;
if my new total is less than 3, I throw again.

Show that, if I have won such a game, the probability that I threw the dart three times is 343/2231.

STEP I 2000 Question 14 (Probability and Statistics)

14 The random variable X is uniformly distributed on the interval $[-1,1]$. Find $\mathrm{E}(X^2)$ and $\mathrm{Var}\,(X^2)$.

A second random variable Y, independent of X, is also uniformly distributed on $[-1,1]$, and $Z=Y-X$. Find $\mathrm{E}(Z^2)$ and show that $\mathrm{Var}\,(Z^2)=7\mathrm{Var}\,(X^2)$.

STEP I 1998 Question 14 (Probability and Statistics)

14 To celebrate the opening of the financial year the finance minister of Genland flings a Slihing, a circular coin of radius a cm, where $0 < a < 1$, onto a large board divided into squares by two sets of parallel lines 2 cm apart. If the coin does not cross any line, or if the coin covers an intersection, the tax on yaks remains unchanged. Otherwise the tax is doubled. Show that, in order to raise most tax, the value of a should be

$$\left(1+\frac{\pi}{4}\right)^{-1}.$$

If, indeed, $a = \left(1+\frac{\pi}{4}\right)^{-1}$ and the tax on yaks is 1 Slihing per yak this year, show that its expected value after n years will have passed is

$$\left(\frac{8+\pi}{4+\pi}\right)^{n}.$$

STEP III 2001 Question 14 (Probability and Statistics)

14 A random variable X is distributed uniformly on $[\,0\,,\,a\,]$. Show that the variance of X is $\frac{1}{12}a^2$.

A sample, X_1 and X_2, of two independent values of the random variable is drawn, and the variance V of the sample is determined. Show that $V = \frac{1}{4}\left(X_1 - X_2\right)^2$, and hence prove that $2V$ is an unbiased estimator of the variance of X.

Find an exact expression for the probability that the value of V is less than $\frac{1}{12}a^2$ and estimate the value of this probability correct to one significant figure.

STEP I 1997 Question 14 (Probability and Statistics)

14 The maximum height X of flood water each year on a certain river is a random variable with density function

$$\mathrm{f}(x) = \begin{cases} \exp(-x) & \text{if } x \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

It costs y megadollars each year to prepare for flood water of height y or less. If $X \leqslant y$ no further costs are incurred but if $X \geqslant y$ the cost of flood damage is $r + s(X - y)$ megadollars where $r, s > 0$. The total cost T megadollars is thus given by

$$T = \begin{cases} y & \text{if } X \leqslant y, \\ y + r + s(X - y) & \text{if } X > y. \end{cases}$$

Show that we can minimise the expected total cost by taking

$$y = \ln(r + s).$$

STEP III 1992 Question 15 (Probability and Statistics)

15 A goat G lies in a square field $OABC$ of side a. It wanders randomly round its field, so that at any time the probability of its being in any given region is proportional to the area of this region. Write down the probability that its distance, R, from O is less than r if $0 < r \leqslant a$, and show that if $r \geqslant a$ the probability is

$$\left(\frac{r^2}{a^2} - 1\right)^{\frac{1}{2}} + \frac{\pi r^2}{4a^2} - \frac{r^2}{a^2}\cos^{-1}\left(\frac{a}{r}\right).$$

Find the median of R and probability density function of R.

The goat is then tethered to the corner O by a chain of length a. Find the conditional probability that its distance from the fence OC is more than $a/2$.