STEP Past Papers by Topic

STEP Topic - Curve sketching

STEP III 1990 Question 7 (Pure)

7 The points $P\,(0,a)$, $Q\,(a,0)$ and $R\,(a,-a)$ lie on the curve C with cartesian equation

$$xy^2 + x^3 + a^2y - a^3 = 0, \qquad \text{where } a > 0.$$

At each of P, Q and R, express y as a Taylor series in h, where h is a small increment in x, as far as the term in h^2. Hence, or otherwise, sketch the shape of C near each of these points.

Show that, if (x, y) lies on C, then

$$4x^4 - 4a^3x - a^4 \leqslant 0.$$

Sketch the graph of $y = 4x^4 - 4a^3 - a^4$.

Given that the y-axis is an asymptote to C, sketch the curve C.

STEP III 1992 Question 4 (Pure)

4 A set of curves S_1 is defined by the equation

$$y = \frac{x}{x-a},$$

where a is a constant which is different for different members of S_1. Sketch on the same axes the curves for which $a = -2, -1, 1$ and 2.

A second of curves S_2 is such that at each intersection between a member of S_2 and a member of S_1 the tangents of the intersecting curves are perpendicular. On the same axes as the already sketched members of S_1, sketch the member of S_2 that passes through the point $(1, -1)$.

Obtain the first order differential equation for y satisfied at all points on all members of S_1 (i.e. an equation connecting x, y and $\mathrm{d}y/\mathrm{d}x$ which does not involve a).

State the relationship between the values of $\mathrm{d}y/\mathrm{d}x$ on two intersecting curves, one from S_1 and one from S_2, at their intersection. Hence show that the differential equation for the curves of S_2 is

$$x = y(y-1)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Find an equation for the member of S_2 that you have sketched.

STEP I 1987 Question 1 (Pure)

1 Find the stationary points of the function f given by

$$\mathrm{f}(x) = \mathrm{e}^{ax}\cos bx, \ (a > 0, b > 0).$$

Show that the values of f at the stationary points with $x > 0$ form a geometric progression with common ratio $-\mathrm{e}^{a\pi/b}$.

Give a rough sketch of the graph of f.

STEP I 1989 Question 7 (Pure)

7 Sketch the curve $y^2 = 1 - |x|$. A rectangle, with sides parallel to the axes, is inscribed within this curve. Show that the largest possible area of the rectangle is $8/\sqrt{27}$.

Find the maximum area of a rectangle similarly inscribed within the curve given by $y^{2m} = (1 - |x|)^n$, where m and n are positive integers, with n odd.

STEP III 1988 Question 1 (Pure)

1 Sketch the graph of

$$y = \frac{x^2 e^{-x}}{1+x},$$

for $-\infty < x < \infty$.

Show that the value of

$$\int_0^\infty \frac{x^2 e^{-x}}{1+x}\,dx$$

lies between 0 and 1.

STEP I 1988 Question 1 (Pure)

1 Sketch the graph of the function h, where

$$\mathrm{h}(x) = \frac{\ln x}{x}, \qquad (x > 0).$$

Hence, or otherwise, find all pairs of distinct positive integers m and n which satisfy the equation

$$n^m = m^n.$$

STEP I 1989 Question 9 (Pure)

9 Sketch the graph of $8y = x^3 - 12x$ for $-4 \leqslant x \leqslant 4$, marking the coordinates of the turning points. Similarly marking the turning points, sketch the corresponding graphs in the (X, Y)-plane, if

$$\begin{array}{lll} \text{(a)} & X = \tfrac{1}{2}x, & Y = y, \\ \text{(b)} & X = x, & Y = \tfrac{1}{2}y, \\ \text{(c)} & X = \tfrac{1}{2}x + 1, & Y = y, \\ \text{(d)} & X = x, & Y = \tfrac{1}{2}y + 1. \end{array}$$

Find values for a, b, c, d such that, if $X = ax + b$, $Y = cy + d$, then the graph in the (X, Y)-plane corresponding to $8y = x^3 - 12x$ has turning points at $(X, Y) = (0, 0)$ and $(X, Y) = (1, 1)$.

STEP II 1989 Question 4 (Pure)

4 The function f is defined by

$$\mathrm{f}(x) = \frac{(x-a)(x-b)}{(x-c)(x-d)} \qquad (x \neq c,\ x \neq d),$$

where a, b, c and d are real and distinct, and $a + d \neq c + d$. Show that

$$\frac{x\mathrm{f}'(x)}{\mathrm{f}(x)} = \left(1 - \frac{a}{x}\right)^{-1} + \left(1 - \frac{b}{x}\right)^{-1} - \left(1 - \frac{c}{x}\right)^{-1} - \left(1 - \frac{d}{x}\right)^{-1},$$

$(x \neq 0, x \neq a, x \neq b)$ and deduce that when $|x|$ is much larger than each of $|a|$, $|b|$, $|c|$ and $|d|$, the gradient of $\mathrm{f}(x)$ has the same sign as $(a + b - c - d)$.

It is given that there is a real value of real value of x for which $\mathrm{f}(x)$ takes the real value z if and only if

$$[(c-d)^2 z + (a-c)(b-d) + (a-d)(b-c)]^2 \geqslant 4(a-c)(b-d)(a-d)(b-c).$$

Describe briefly a method by which this result could be proved, but do not attempt to prove it.

Given that $a < b$ and $a < c < d$, make sketches of the graph of f in the four distinct cases which arise, indicating the cases for which the range of f is not the whole of $\mathbb{R}$.

STEP II 1988 Question 6 (Pure)

6 Show that the following functions are positive when x is positive:

(i) $x - \tanh x$

(ii) $x \sinh x - 2\cosh x + 2$

(iii) $2x\cosh 2x - 3\sinh 2x + 4x.$

The function f is defined for $x > 0$ by

$$\mathrm{f}(x) = \frac{x(\cosh x)^{\frac{1}{3}}}{\sinh x}.$$

Show that $\mathrm{f}(x)$ has no turning points when $x > 0$, and sketch $\mathrm{f}(x)$ for $x > 0$.

STEP III 1989 Question 4 (Pure)

4 Sketch the curve whose cartesian equation is

$$y = \frac{2x(x^2 - 5)}{x^2 - 4},$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence, or otherwise, determine (giving reasons) the number of real roots of the following equations:

(i) $4x^2(x^2 - 5) = (5x - 2)(x^2 - 4);$

(ii) $4x^2(x^2 - 5)^2 = (x^2 - 4)^2(x^2 + 1);$

(iii) $4z^2(z - 5)^2 = (z - 4)^2(z + 1).$

STEP II 1992 Question 6 (Pure)

6 Sketch the graphs of $y = \sec x$ and $y = \ln(2 \sec x)$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. Show graphically that the equation

$$kx = \ln(2 \sec x)$$

has no solution with $0 \leqslant x < \frac{1}{2}\pi$ if k is a small positive number but two solutions if k is large.

Explain why there is a number k_0 such that

$$k_0 x = \ln(2 \sec x)$$

has exactly one solution with $0 \leqslant x < \frac{1}{2}\pi$. Let x_0 be this solution, so that $0 \leqslant x_0 < \frac{1}{2}\pi$ and $k_0 x_0 = \ln(2 \sec x)$. Show that

$$x_0 = \cot x_0 \ln(2 \sec x_0).$$

Use any appropriate method to find x_0 correct to two decimal places. Hence find an approximate value for k_0.

STEP II Specimen Question 4 (Pure)

4 **(i)** Show that

$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x),$$

when principal values only are considered.

(ii) Show that

$$\sinh^{-1}(\tan y) = \tanh^{-1}(\sin y),$$

when $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

Sketch the graphs of $\sinh^{-1}(\tan y)$ and $\tanh^{-1}(\sin y)$ in the interval $-\pi < y < \pi$ and find the relationship between the two expressions when $\frac{1}{2}\pi < y < \pi$.

STEP II 1990 Question 3 (Pure)

3 Sketch the curves given by

$$y = x^3 - 2bx^2 + c^2x,$$

where b and c are non-negative, in the cases:

$$\textbf{(i)}\ \ 2b < c\sqrt{3}, \quad \textbf{(ii)}\ \ 2b = c\sqrt{3} \neq 0, \quad \textbf{(iii)}\ \ c\sqrt{3} < 2b < 2c, \quad \textbf{(iv)}\ \ b = c \neq 0,$$
$$\textbf{(v)}\ \ b > c > 0, \quad \textbf{(vi)}\ \ \ c = 0, b \neq 0, \quad \textbf{(vii)}\ \ c = b = 0.$$

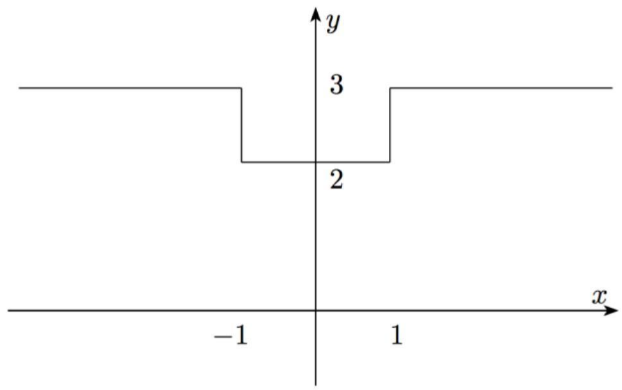
Sketch also the curves given by $y^2 = x^3 - 2bx^2 + c^2x$ in the cases **(i)**, **(v)** and **(vii)**.

STEP I 1993 Question 6 (Pure)

6 Let $N = 10^{100}$. The graph of

$$f(x) = \frac{x^N}{1 + x^N} + 2$$

for $-3 \leqslant x \leqslant 3$ is sketched in the following diagram.



Explain the main features of the sketch.

Sketch the graphs for $-3 \leqslant x \leqslant 3$ of the two functions

$$g(x) = \frac{x^{N+1}}{1 + x^N}$$

and

$$h(x) = 10^N \sin(10^{-N} x).$$

In each case explain briefly the main features of your sketch.

STEP III 2000 Question 1 (Pure)

1 Sketch on the same axes the two curves C_1 and C_2, given by

$$\begin{aligned} C_1: \quad & xy &= 1\,, \\ C_2: \quad & x^2 - y^2 &= 2\,. \end{aligned}$$

The curves intersect at P and Q. Given that the coordinates of P are (a, b) (which you need not evaluate), write down the coordinates of Q in terms of a and b.

The tangent to C_1 through P meets the tangent to C_2 through Q at the point M, and the tangent to C_2 through P meets the tangent to C_1 through Q at N. Show that the coordinates of M are $(-b, a)$ and write down the coordinates of N.

Show that $PMQN$ is a square.

STEP I 2002 Question 4 (Pure)

4 Give a sketch of the curve $y = \dfrac{1}{1+x^2}$, for $x \geqslant 0$.

Find the equation of the line that intersects the curve at $x = 0$ and is tangent to the curve at some point with $x > 0$. Prove that there are no further intersections between the line and the curve. Draw the line on your sketch.

By considering the area under the curve for $0 \leqslant x \leqslant 1$, show that $\pi > 3$.

Show also, by considering the volume formed by rotating the curve about the y axis, that $\ln 2 > 2/3$.

[**Note**: $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$.]

STEP III 2005 Question 1 (Pure)

1 Show that $\sin A = \cos B$ if and only if $A = (4n+1)\dfrac{\pi}{2} \pm B$ for some integer n.

Show also that $\left|\sin x \pm \cos x\right| \leqslant \sqrt{2}$ for all values of x and deduce that there are no solutions to the equation $\sin(\sin x) = \cos(\cos x)$.

Sketch, on the same axes, the graphs of $y = \sin(\sin x)$ and $y = \cos(\cos x)$. Sketch, not on the previous axes, the graph of $y = \sin(2\sin x)$.

STEP I 2015 Question 1 (Pure)

1 **(i)** Sketch the curve $y = e^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $e^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k.

You may assume that $x^n e^x \to 0$ as $x \to -\infty$ for any integer n.

(ii) Sketch the curve $y = e^{x^2}(2x^4 - 5x^2 + 2)$.

STEP I 2000 Question 6 (Pure)

6 Show that

$$x^2 - y^2 + x + 3y - 2 = (x - y + 2)(x + y - 1)$$

and hence, or otherwise, indicate by means of a sketch the region of the x-y plane for which

$$x^2 - y^2 + x + 3y > 2.$$

Sketch also the region of the x-y plane for which

$$x^2 - 4y^2 + 3x - 2y < -2.$$

Give the coordinates of a point for which both inequalities are satisfied or explain why no such point exists.

STEP III 1997 Question 2 (Pure)

2 Let

$$f(t) = \frac{\ln t}{t} \quad \text{for } t > 0.$$

Sketch the graph of $f(t)$ and find its maximum value. How many positive values of t correspond to a given value of $f(t)$?

Find how many positive values of y satisfy $x^y = y^x$ for a given positive value of x. Sketch the set of points (x, y) which satisfy $x^y = y^x$ with $x, y > 0$.

STEP III 2006 Question 1 (Pure)

1 Sketch the curve with cartesian equation

$$y = \frac{2x(x^2-5)}{x^2-4}$$

and give the equations of the asymptotes and of the tangent to the curve at the origin.

Hence determine the number of real roots of the following equations:

(i) $3x(x^2-5) = (x^2-4)(x+3)\,;$

(ii) $4x(x^2-5) = (x^2-4)(5x-2)\,;$

(iii) $4x^2(x^2-5)^2 = (x^2-4)^2(x^2+1)\,.$

STEP I 2004 Question 2 (Pure)

2 The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}] = 4$ and $[5] = 5$.

(i) Sketch the graph of $y = \sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\, \mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\int_0^a 2^{[x]}\, \mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\int_0^a 2^{[x]}\, \mathrm{d}x$ when a is positive but not an integer.

STEP III 2000 Question 4 (Pure)

4 The function $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = \frac{x(x-2)(x-a)}{x^2-1}.$$

Prove algebraically that the line $y = x + c$ intersects the curve $y = \mathrm{f}(x)$ if $|a| \geqslant 1$, but there are values of c for which there are no points of intersection if $|a| < 1$.

Find the equation of the oblique asymptote of the curve $y = \mathrm{f}(x)$. Sketch the graph in the two cases (i) $a < -1$; and (ii) $-1 < a < -\frac{1}{2}$. (You need not calculate the turning points.)

STEP I 2012 Question 2 (Pure)

2 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

STEP II 2001 Question 2 (Pure)

2 Sketch the graph of the function $[x/N]$, for $0 < x < 2N$, where the notation $[y]$ means the integer part of y. (Thus $[2.9] = 2,\ [4] = 4$.)

(i) Prove that

$$\sum_{k=1}^{2N}(-1)^{[k/N]}k = 2N - N^2.$$

(ii) Let

$$S_N = \sum_{k=1}^{2N}(-1)^{[k/N]}2^{-k}.$$

Find S_N in terms of N and determine the limit of S_N as $N \to \infty$.

STEP III 1998 Question 1 (Pure)

1 Let

$$f(x) = \sin^2 x + 2\cos x + 1$$

for $0 \leqslant x \leqslant 2\pi$. Sketch the curve $y = f(x)$, giving the coordinates of the stationary points. Now let

$$g(x) = \frac{af(x) + b}{cf(x) + d} \qquad ad \neq bc\,,\ d \neq -3c\,,\ d \neq c\,.$$

Show that the stationary points of $y = g(x)$ occur at the same values of x as those of $y = f(x)$, and find the corresponding values of $g(x)$.

Explain why, if $d/c < -3$ or $d/c > 1$, $|g(x)|$ cannot be arbitrarily large.

STEP I 1999 Question 4 (Pure)

4 Sketch the following subsets of the x-y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x-1| + |y-1| \leqslant 1$;

(iii) $|x-1| - |y+1| \leqslant 1$;

(iv) $|x|\,|y-2| \leqslant 1$.

STEP I 1997 Question 8 (Pure)

8 By considering the maximum of $\ln x - x \ln a$, or otherwise, show that the equation $x = a^x$ has no real roots if $a > e^{1/e}$.

How many real roots does the equation have if $0 < a < 1$? Justify your answer.

STEP I 2002 Question 2 (Pure)

2 Let $f(x) = x^m(x-1)^n$, where m and n are both integers greater than 1. Show that the curve $y = f(x)$ has a stationary point with $0 < x < 1$. By considering $f''(x)$, show that this stationary point is a maximum if n is even and a minimum if n is odd.

Sketch the graphs of $f(x)$ in the four cases that arise according to the values of m and n.

STEP I 2007 Question 8 (Pure)

8 A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16)\ , \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2\,,\,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

STEP III 2012 Question 3 (Pure)

3 It is given that the two curves

$$y = 4 - x^2 \quad \text{and} \quad mx = k - y^2\,,$$

where $m > 0$, touch exactly once.

(i) In each of the following four cases, sketch the two curves on a single diagram, noting the coordinates of any intersections with the axes:

(a) $k < 0$;

(b) $0 < k < 16$, $k/m < 2$;

(c) $k > 16$, $k/m > 2$;

(d) $k > 16$, $k/m < 2$.

(ii) Now set $m = 12$.
Show that the x-coordinate of any point at which the two curves meet satisfies

$$x^4 - 8x^2 + 12x + 16 - k = 0\,.$$

Let a be the value of x at the point where the curves touch. Show that a satisfies

$$a^3 - 4a + 3 = 0$$

and hence find the three possible values of a.

Derive also the equation

$$k = -4a^2 + 9a + 16\,.$$

Which of the four sketches in part (i) arise?

STEP II 2012 Question 5 (Pure)

5 **(i)** Sketch the curve $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{(x-a)^2 - 1} \qquad (x \neq a \pm 1),$$

and a is a constant.

(ii) The function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = \frac{1}{\big((x-a)^2 - 1\big)\big((x-b)^2 - 1\big)} \qquad (x \neq a \pm 1,\ x \neq b \pm 1),$$

where a and b are constants, and $b > a$. Sketch the curves $y = \mathrm{g}(x)$ in the two cases $b > a + 2$ and $b = a + 2$, finding the values of x at the stationary points.

STEP III 1999 Question 2 (Pure)

2 **(i)** Let $\mathrm{f}(x) = (1 + x^2)\mathrm{e}^x$. Show that $\mathrm{f}'(x) \geqslant 0$ and sketch the graph of $\mathrm{f}(x)$. Hence, or otherwise, show that the equation

$$(1 + x^2)\mathrm{e}^x = k,$$

where k is a constant, has exactly one real root if $k > 0$ and no real roots if $k \leqslant 0$.

(ii) Determine the number of real roots of the equation

$$(\mathrm{e}^x - 1) - k\tan^{-1} x = 0$$

in the cases (a) $0 < k \leqslant 2/\pi$ and (b) $2/\pi < k < 1$.

STEP II 2015 Question 4 (Pure)

4 **(i)** The continuous function f is defined by

$$\tan \mathrm{f}(x) = x \qquad (-\infty < x < \infty)$$

and $\mathrm{f}(0) = \pi$. Sketch the curve $y = \mathrm{f}(x)$.

(ii) The continuous function g is defined by

$$\tan \mathrm{g}(x) = \frac{x}{1+x^2} \qquad (-\infty < x < \infty)$$

and $\mathrm{g}(0) = \pi$. Sketch the curves $y = \dfrac{x}{1+x^2}$ and $y = \mathrm{g}(x)$.

(iii) The continuous function h is defined by $\mathrm{h}(0) = \pi$ and

$$\tan \mathrm{h}(x) = \frac{x}{1-x^2} \qquad (x \neq \pm 1).$$

(The values of $\mathrm{h}(x)$ at $x = \pm 1$ are such that $\mathrm{h}(x)$ is continuous at these points.)

Sketch the curves $y = \dfrac{x}{1-x^2}$ and $y = \mathrm{h}(x)$.

STEP II 1999 Question 7 (Pure)

7 The curve C has equation

$$y = \frac{x}{\sqrt{x^2 - 2x + a}}\,,$$

where the square root is positive. Show that, if $a > 1$, then C has exactly one stationary point.

Sketch C when **(i)** $a = 2$ and **(ii)** $a = 1$.

STEP III 1998 Question 7 (Pure)

7 Sketch the graph of $\mathrm{f}(s) = \mathrm{e}^s(s-3)+3$ for $0 \leqslant s < \infty$. Taking $\mathrm{e} \approx 2.7$, find the smallest positive integer, m, such that $\mathrm{f}(m) > 0$.

Now let

$$\mathrm{b}(x) = \frac{x^3}{\mathrm{e}^{x/T}-1}$$

where T is a positive constant. Show that $\mathrm{b}(x)$ has a single turning point in $0 < x < \infty$. By considering the behaviour for small x and for large x, sketch $\mathrm{b}(x)$ for $0 \leqslant x < \infty$.

Let

$$\int_0^\infty \mathrm{b}(x)\,\mathrm{d}x = B,$$

which may be assumed to be finite. Show that $B = KT^n$ where K is a constant, and n is an integer which you should determine.

Given that $B \approx 2\displaystyle\int_0^{Tm} \mathrm{b}(x)\,\mathrm{d}x$, use your graph of $\mathrm{b}(x)$ to find a rough estimate for K.

STEP III 2003 Question 3 (Pure)

3 If m is a positive integer, show that $(1+x)^m + (1-x)^m \neq 0$ for any real x.

The function f is defined by

$$\mathrm{f}(x) = \frac{(1+x)^m - (1-x)^m}{(1+x)^m + (1-x)^m} .$$

Find and simplify an expression for $\mathrm{f}'(x)$.

In the case $m = 5$, sketch the curves $y = \mathrm{f}(x)$ and $y = \dfrac{1}{\mathrm{f}(x)}$.

STEP III 2002 Question 3 (Pure)

3 Let

$$f(x) = a\sqrt{x} - \sqrt{x-b}\,,$$

where $x \geqslant b > 0$ and $a > 1$. Sketch the graph of $f(x)$. Hence show that the equation $f(x) = c$, where $c > 0$, has no solution when $c^2 < b\left(a^2 - 1\right)$. Find conditions on c^2 in terms of a and b for the equation to have exactly one or exactly two solutions.

Solve the equations

(i) $3\sqrt{x} - \sqrt{x-2} = 4\,,$

(ii) $3\sqrt{x} - \sqrt{x-3} = 5\,.$

STEP II 1997 Question 7 (Pure)

7 Let

$$y^2 = x^2(a^2 - x^2),$$

where a is a real constant. Find, in terms of a, the maximum and minimum values of y. Sketch carefully on the same axes the graphs of y in the cases $a = 1$ and $a = 2$.

STEP I 2010 Question 2 (Pure)

2 The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

STEP II 2004 Question 3 (Pure)

3 The curve C has equation

$$y = x(x+1)(x-2)^4.$$

Determine the coordinates of all the stationary points of C and the nature of each. Sketch C.

In separate diagrams draw sketches of the curves whose equations are:

(i) $y^2 = x(x+1)(x-2)^4$;

(ii) $y = x^2(x^2+1)(x^2-2)^4$.

STEP III 2004 Question 2 (Pure)

2 The equation of a curve is $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x - 4 - \frac{16\,(2x+1)^2}{x^2\,(x-4)}\ .$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $\mathrm{f}(x) = 0$ has a double root.

(iii) Sketch the curve.

STEP I 2013 Question 5 (Pure)

5 The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \quad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45°$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

STEP II 2013 Question 1 (Pure)

1 **(i)** Find the value of m for which the line $y = mx$ touches the curve $y = \ln x$.

If instead the line intersects the curve when $x = a$ and $x = b$, where $a < b$, show that $a^b = b^a$. Show by means of a sketch that $a < \mathrm{e} < b$.

(ii) The line $y = mx + c$, where $c > 0$, intersects the curve $y = \ln x$ when $x = p$ and $x = q$, where $p < q$. Show by means of a sketch, or otherwise, that $p^q > q^p$.

(iii) Show by means of a sketch that the straight line through the points $(p, \ln p)$ and $(q, \ln q)$, where $\mathrm{e} \leqslant p < q$, intersects the y-axis at a positive value of y. Which is greater, π^{e} or e^{π}?

(iv) Show, using a sketch or otherwise, that if $0 < p < q$ and $\dfrac{\ln q - \ln p}{q - p} = \mathrm{e}^{-1}$, then $q^p > p^q$.

STEP II 2014 Question 7 (Pure)

7 **(i)** The function f is defined by $\mathrm{f}(x) = |x-a| + |x-b|$, where $a < b$. Sketch the graph of $\mathrm{f}(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $\mathrm{g}(x)$, where $\mathrm{g}(x) = |2x - a - b|$.

What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $(b, \mathrm{f}(b))$ and $(a, \mathrm{f}(a))$?

(ii) Show graphically that the equation

$$|x-a| + |x-b| = |x-c|\,,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x-a| + |x-b| = |x-c| + |x-d|\,,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

STEP I 2014 Question 3 (Pure)

3 The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2 \, \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x \right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.
