STEP Past Papers by Topic

STEP Topic - Differential equation

STEP III Specimen Question 11 (Mechanics)

11 Two identical snowploughs plough the same stretch of road in the same direction. The first starts at time $t = 0$ when the depth of snow is h metres, and the second starts from the same point T seconds later. Snow falls so that the depth of snow increases at a constant rate of k ms^{-1}. It may be assumed that each snowplough moves at a speed equal to $k/(\alpha z)$ ms^{-1} with α a constant, where z is the depth of snow it is ploughing, and that it clears all the snow.

(i) Show that the time taken for the first snowplough to travel x metres is

$$(\mathrm{e}^{\alpha x} - 1)\frac{h}{k} \text{ seconds.}$$

(ii) Show that at time $t > T$, the second snowplough has moved y metres, where t satisfies

$$\frac{1}{\alpha}\frac{\mathrm{d}t}{\mathrm{d}y} = t - (\mathrm{e}^{\alpha y} - 1)\frac{h}{k}.$$

Verify that the required solution of this equation is

$$t = (\mathrm{e}^{\alpha y} - 1)\frac{h}{k} + \left(T - \frac{\alpha h y}{k}\right)\mathrm{e}^{\alpha y}$$

and deduce that the snowploughs collide when they have moved a distance $kT/(\alpha h)$ metres.

STEP III 1989 Question 8 (Pure)

8 Given that

$$\frac{dx}{dt} = 4(x-y) \qquad \text{and} \qquad \frac{dy}{dt} = x - 12(e^{2t} + e^{-2t}),$$

obtain a differential equation for x which does not contain y. Hence, or otherwise, find x and y in terms of t given that $x = y = 0$ when $t = 0$.

STEP II 1988 Question 8 (Pure)

8 In a crude model of population dynamics of a community of aardvarks and buffaloes, it is assumed that, if the numbers of aardvarks and buffaloes in any year are A and B respectively, then the numbers in the following year at $\frac{1}{4}A + \frac{3}{4}B$ and $\frac{3}{2}B - \frac{1}{2}A$ respectively. It does not matter if the model predicts fractions of animals, but a non-positive number of buffaloes means that the species has become extinct, and the model ceases to apply. Using matrices or otherwise, show that the ratio of the number of aardvarks to the number of buffaloes can remain the same each year, provided it takes one of two possible values.

Let these two possible values be x and y, and let the numbers of aardvarks and buffaloes in a given year be a and b respectively. By writing the vector (a, b) as a linear combination of the vectors $(x, 1)$ and $(y, 1)$, or otherwise, show how the numbers of aardvarks and buffaloes in subsequent years may be found. On a sketch of the a-b plane, mark the regions which correspond to the following situations

(i) an equilibrium population is reached as time $t \to \infty$;

(ii) buffaloes become extinct after a finite time;

(iii) buffaloes approach extinction as $t \to \infty$.

STEP III 1990 Question 8 (Pure)

8 Let P, Q and R be functions of x. Prove that, for any function y of x, the function

$$Py'' + Qy' + Ry$$

can be written in the form $\dfrac{\mathrm{d}}{\mathrm{d}x}(py' + qy)$, where p and q are functions of x, if and only if $P'' - Q' + R = 0$.

Solve the differential equation

$$(x - x^4)y'' + (1 - 7x^3)y' - 9x^2y = (x^3 + 3x)\mathrm{e}^x,$$

given that when $x = 2, y = 2\mathrm{e}^2$ and $y' = 0$.

STEP I 1987 Question 3 (Pure)

3 By substituting $y(x) = xv(x)$ in the differential equation

$$x^3 \frac{\mathrm{d}v}{\mathrm{d}x} + x^2 v = \frac{1 + x^2 v^2}{(1 + x^2)\, v},$$

or otherwise, find the solution $v(x)$ that satisfies $v = 1$ when $x = 1$.

What value does this solution approach when x becomes large?

STEP I 1993 Question 9 (Pure)

9 In the manufacture of Grandma's Home Made Ice-cream, chemicals A and B pour at constant rates a and $b - a$ litres per second $(0 < a < b)$ into a mixing vat which mixes the chamicals rapidly and empties at a rate b litres per second into a second mixing vat. At time $t = 0$ the first vat contains K litres of chemical B only. Show that the volume $V(t)$ (in litres) of the chemical A in the first vat is governed by the differential equation

$$\dot{V}(t) = -\frac{bV(t)}{K} + a,$$

and that

$$V(t) = \frac{aK}{b}(1 - \mathrm{e}^{-bt/K})$$

for $t \geqslant 0$.

The second vat also mixes chemicals rapidly and empties at the rate of b litres per second. If at time $t = 0$ it contains L litres of chemical C only (where $L \neq K$), how many litres of chemical A will it contain at a later time t?

STEP III 1988 Question 13 (Mechanics)

13 A goalkeeper stands on the goal-line and kicks the football directly into the wind, at an angle α to the horizontal. The ball has mass m and is kicked with velocity $\mathbf{v}_0$. The wind blows horizontally with constant velocity $\mathbf{w}$ and the air resistance on the ball is mk times its velocity relative to the wind velocity, where k is a positive constant. Show that the equation of motion of the ball can be written in the form

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + k\mathbf{v} = \mathbf{g} + k\mathbf{w},$$

where $\mathbf{v}$ is the ball's velocity relative to the ground, and $\mathbf{g}$ is the acceleration due to gravity.

By writing down horizontal and vertical equations of motion for the ball, or otherwise, find its position at time t after it was kicked.

On the assumption that the goalkeeper moves out of the way, show that if $\tan\alpha = |\mathbf{g}|/(k|\mathbf{w}|)$, then the goalkeeper scores an own goal.

STEP I Specimen Question 3 (Pure)

3 Using the substitution $z = \frac{\mathrm{d}y}{\mathrm{d}x} - y$, or otherwise, solve the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\mathrm{e}^x,$$

given that

$$y = 1 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \text{when } x = 0.$$

STEP III Specimen Question 7 (Pure)

7 Show that the differential equation

$$x^2y'' + (x-2)(xy'-y) = 0$$

has a solution proportional to x^α for some α. By making the substitution $y = x^\alpha v$, or otherwise, find the general solution of this equation.

STEP II 1989 Question 7 (Pure)

7 By means of the substitution x^α, where α is a suitably chosen constant, find the general solution for $x > 0$ of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - b\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2b+1}y = 0,$$

where b is a constant and $b > -1$.

Show that, if $b > 0$, there exist solutions which satisfy $y \to 1$ and $\mathrm{d}y/\mathrm{d}x \to 0$ as $x \to 0$, but that these conditions do not determine a unique solution. For what values of b do these conditions determine a unique solution?

STEP II 1993 Question 3 (Pure)

3 **(i)** Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y - 3y^2 = -2$$

by making the substitution $y = -\frac{1}{3u}\frac{\mathrm{d}u}{\mathrm{d}x}$.

(ii) Solve the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy + x^2y^2 = 1$$

by making the substitution

$$y = \frac{1}{x} + \frac{1}{v},$$

where v is a function of x.

STEP II 2007 Question 6 (Pure)

6 **(i)** Differentiate $\ln\left(x+\sqrt{3+x^2}\right)$ and $x\sqrt{3+x^2}$ and simplify your answers.

Hence find $\int \sqrt{3+x^2}\,\mathrm{d}x$.

(ii) Find the two solutions of the differential equation

$$3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

that satisfy $y = 0$ when $x = 1$.

STEP III 2012 Question 1 (Pure)

1 Given that $z = y^n \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).$$

(i) Use the above result to show that the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

that satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

STEP III 1997 Question 6 (Pure)

6 Suppose that y_n satisfies the equations

$$(1-x^2)\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - x\frac{\mathrm{d}y_n}{\mathrm{d}x} + n^2 y_n = 0,$$

$$y_n(1) = 1, \quad y_n(x) = (-1)^n y_n(-x).$$

If $x = \cos\theta$, show that

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}\theta^2} + n^2 y_n = 0,$$

and hence obtain y_n as a function of θ. Deduce that for $|x| \leqslant 1$

$$y_0 = 1, \quad y_1 = x,$$

$$y_{n+1} - 2xy_n + y_{n-1} = 0.$$

STEP III 1988 Question 7 (Pure)

7 For $n = 0, 1, 2, \ldots$, the functions y_n satisfy the differential equation

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}x^2} - \omega^2 x^2 y_n = -(2n+1)\omega y_n,$$

where ω is a positive constant, and $y_n \to 0$ and $\mathrm{d}y_n/\mathrm{d}x \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = \mathrm{e}^{-\lambda x^2} \qquad \text{and} \qquad y_1(x) = x\mathrm{e}^{-\lambda x^2}$$

for some constant λ, to be determined.

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y_m \frac{\mathrm{d}y_n}{\mathrm{d}x} - y_n \frac{\mathrm{d}y_m}{\mathrm{d}x}\right) = 2(m-n)\omega y_m y_n,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x)\,\mathrm{d}x = 0.$$

STEP II 2001 Question 8 (Pure)

8 The function f satisfies $\mathrm{f}(x+1) = \mathrm{f}(x)$ and $\mathrm{f}(x) > 0$ for all x.

(i) Give an example of such a function.

(ii) The function F satisfies

$$\frac{\mathrm{dF}}{\mathrm{d}x} = \mathrm{f}(x)$$

and $\mathrm{F}(0) = 0$. Show that $\mathrm{F}(n) = n\mathrm{F}(1)$, for any positive integer n.

(iii) Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{f}(x)y = 0$$

that satisfies $y = 1$ when $x = 0$. Show that $y(n) \to 0$ as $n \to \infty$, where $n = 1, 2, 3, \ldots$

STEP I 1995 Question 8 (Pure)

8 Find functions f, g and h such that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{f}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{g}(x)y = \mathrm{h}(x) \qquad (*)$$

is satisfied by all three of the solutions $y = x$, $y = 1$ and $y = x^{-1}$ for $0 < x < 1$.

If f, g and h are the functions you have found in the first paragraph, what condition must the real numbers a, b and c satisfy in order that

$$y = ax + b + \frac{c}{x}$$

should be a solution of $(*)$?

STEP III 2001 Question 7 (Pure)

7 Sketch the graph of the function $\ln x - \frac{1}{2}x^2$.

Show that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - 1}$$

describes a family of parabolas each of which passes through the points $(1\ ,\ 0)$ and $(-1\ ,\ 0)$ and has its vertex on the y–axis.

Hence find the equation of the curve that passes through the point $(1\ ,\ 1)$ and intersects each of the above parabolas orthogonally. Sketch this curve.

[Two curves intersect *orthogonally* if their tangents at the point of intersection are perpendicular.]

STEP II 1996 Question 8 (Pure)

8 Suppose that

$$f''(x) + f(-x) = x + 3\cos 2x$$

and $f(0) = 1$, $f'(0) = -1$. If $g(x) = f(x) + f(-x)$, find $g(0)$ and show that $g'(0) = 0$. Show that

$$g''(x) + g(x) = 6\cos 2x,$$

and hence find $g(x)$.

Similarly, if $h(x) = f(x) - f(-x)$, find $h(x)$ and show that

$$f(x) = 2\cos x - \cos 2x - x.$$

STEP III 1995 Question 3 (Pure)

3 What is the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + x = 0$$

for each of the cases: (i) $k > 1$; (ii) $k = 1$; (iii) $0 < k < 1$?

In case (iii) the equation represents damped simple harmonic motion with damping factor k. Let $x(0) = 0$ and let $x_1, x_2, \ldots, x_n, \ldots$ be the sequence of successive maxima and minima, so that if x_n is a maximum then x_{n+1} is the next minimum. Show that $|x_{n+1}/x_n|$ takes a value α which is independent of n, and that

$$k^2 = \frac{(\ln \alpha)^2}{\pi^2 + (\ln \alpha)^2}.$$

STEP I 1999 Question 7 (Pure)

7 Show that $\sin(k\sin^{-1}x)$, where k is a constant, satisfies the differential equation

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + k^2y = 0. \qquad (*)$$

In the particular case when $k = 3$, find the solution of equation $(*)$ of the form

$$y = Ax^3 + Bx^2 + Cx + D,$$

that satisfies $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3$ at $x = 0$.

Use this result to express $\sin 3\theta$ in terms of powers of $\sin\theta$.

STEP III 2003 Question 8 (Pure)

8 **(i)** Show that the gradient at a point $(x\,,\,y)$ on the curve

$$(y+2x)^3\,(y-4x)=c\;,$$

where c is a constant, is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{16x-y}{2y-5x}\;.$$

(ii) By considering the derivative with respect to x of $(y+ax)^n\,(y+bx)$, or otherwise, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{10x-4y}{3x-y}\;.$$

STEP I 1996 Question 7 (Pure)

7 **(i)** At time $t = 0$ a tank contains one unit of water. Water flows out of the tank at a rate proportional to the amount of water in the tank. The amount of water in the tank at time t is y. Show that there is a constant $b < 1$ such that $y = b^t$.

(ii) Suppose instead that the tank contains one unit of water at time $t = 0$, but that in addition to water flowing out as described, water is added at a steady rate $a > 0$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y \ln b = a,$$

and hence find y in terms of a, b and t.

STEP II 2001 Question 5 (Pure)

5 The curve C_1 passes through the origin in the x–y plane and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1 - x^2)\mathrm{e}^{-x^2}.$$

Show that C_1 has a minimum point at the origin and a maximum point at $\left(1, \frac{1}{2}\,\mathrm{e}^{-1}\right)$. Find the coordinates of the other stationary point. Give a rough sketch of C_1.

The curve C_2 passes through the origin and its gradient is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(1 - x^2)\mathrm{e}^{-x^3}.$$

Show that C_2 has a minimum point at the origin and a maximum point at $(1, k)$, where $k > \frac{1}{2}\,\mathrm{e}^{-1}$. (You need not find k.)

STEP I 2005 Question 8 (Pure)

8 Show that, if $y^2 = x^k \mathrm{f}(x)$, then $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}\dfrac{\mathrm{df}}{\mathrm{d}x}$.

(i) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

(ii) Find the solution of the differential equation

$$2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln(x) - xy^2$$

for which $y = 1$ when $x = 1$.

STEP III 2012 Question 7 (Pure)

7 A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time t of the drug in the blood and the brain respectively are $y(t)$ and $z(t)$. These satisfy

$$\dot{y} = -2(y - z), \qquad \dot{z} = -\dot{y} - 3z,$$

where the dot denotes differentiation with respect to t.

Obtain a second order differential equation for y and hence derive the solution

$$y = A\mathrm{e}^{-t} + B\mathrm{e}^{-6t}, \qquad z = \tfrac{1}{2}A\mathrm{e}^{-t} - 2B\mathrm{e}^{-6t},$$

where A and B are arbitrary constants.

(i) Obtain the solution that satisfies $z(0) = 0$ and $y(0) = 5$. The quantity of the drug in the brain for this solution is denoted by $z_1(t)$.

(ii) Obtain the solution that satisfies $z(0) = z(1) = c$, where c is a given constant. The quantity of the drug in the brain for this solution is denoted by $z_2(t)$.

(iii) Show that for $0 \leqslant t \leqslant 1$,

$$z_2(t) = \sum_{n=-\infty}^{0} z_1(t - n),$$

provided c takes a particular value that you should find.

STEP III 1999 Question 8 (Pure)

8 The function $y(x)$ is defined for $x \geqslant 0$ and satisfies the conditions

$$y = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \text{at } x = 0.$$

When x is in the range $2(n-1)\pi < x < 2n\pi$, where n is a positive integer, $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + n^2 y = 0.$$

Both y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ are continuous at $x = 2n\pi$ for $n = 0,\ 1,\ 2,\ \ldots$.

(i) Find $y(x)$ for $0 \leqslant x \leqslant 2\pi$.

(ii) Show that $y(x) = \frac{1}{2}\sin 2x$ for $2\pi \leqslant x \leqslant 4\pi$, and find $y(x)$ for all $x \geqslant 0$.

(iii) Show that

$$\int_0^\infty y^2 \, \mathrm{d}x = \pi \sum_{n=1}^{\infty} \frac{1}{n^2}\,.$$

STEP II 2000 Question 8 (Pure)

8 **(i)** Let y be the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\mathrm{e}^{-x^2}(y+3)^{\frac{1}{2}} = 0 \qquad (x \geqslant 0),$$

that satisfies the condition $y = 6$ when $x = 0$. Find y in terms of x and show that $y \to 1$ as $x \to \infty$.

(ii) Let y be any solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x\mathrm{e}^{6x^2}(y+3)^{1-k} = 0 \qquad (x \geqslant 0).$$

Find a value of k such that, as $x \to \infty$, $\mathrm{e}^{-3x^2}y$ tends to a finite non-zero limit, which you should determine.

[The approximations, valid for small θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ may be assumed.]

STEP II 2002 Question 8 (Pure)

8 Find y in terms of x, given that:

$$\text{for } x < 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -y \quad \text{and} \quad y = a \text{ when } x = -1\,;$$
$$\text{for } x > 0\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y \quad \text{and} \quad y = b \text{ when } x = 1\,.$$

Sketch a solution curve. Determine the condition on a and b for the solution curve to be continuous (that is, for there to be no 'jump' in the value of y) at $x = 0$.

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = |\mathrm{e}^x - 1|\, y$$

given that $y = \mathrm{e}^{\mathrm{e}}$ when $x = 1$ and that y is continuous at $x = 0$. Write down the following limits:

(i) $\displaystyle\lim_{x \to +\infty} y \exp(-\mathrm{e}^x)\,;$ **(ii)** $\displaystyle\lim_{x \to -\infty} y\mathrm{e}^{-x}\,.$

STEP III 2004 Question 8 (Pure)

8 Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)y + \frac{\mathrm{g}(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$.

Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}.$$

Determine the solution curves of this equation which pass through $(1,1)$, $(2,2)$ and $(4,4)$ and sketch graphs of all three curves on the same axes.

STEP III 2002 Question 6 (Pure)

6 Find all the solution curves of the differential equation

$$y^4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^4 = \left(y^2-1\right)^2$$

that pass through either of the points

(i) $(0,\ \frac{1}{2}\sqrt{3})$,

(ii) $(0,\ \frac{1}{2}\sqrt{5})$.

Show also that $y = 1$ and $y = -1$ are solutions of the differential equation. Sketch all these solution curves on a single set of axes.

STEP I 2012 Question 8 (Pure)

8 **(i)** Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad (x \neq 0).$$

STEP III 2008 Question 6 (Pure)

6 In this question, p denotes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Given that

$$y = p^2 + 2xp\,,$$

show by differentiating with respect to x that

$$\frac{\mathrm{d}x}{\mathrm{d}p} = -2 - \frac{2x}{p}.$$

Hence show that $x = -\frac{2}{3}p + Ap^{-2}$, where A is an arbitrary constant.

Find y in terms of x if $p = -3$ when $x = 2$.

(ii) Given instead that

$$y = 2xp + p\ln p\,,$$

and that $p = 1$ when $x = -\frac{1}{4}$, show that $x = -\frac{1}{2}\ln p - \frac{1}{4}$ and find y in terms of x.

STEP III 2006 Question 7 (Pure)

7 **(i)** Solve the equation $u^2 + 2u\sinh x - 1 = 0$ giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = \mathrm{f}(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

STEP II 1999 Question 9 (Mechanics)

9 In the Z–universe, a star of mass M suddenly blows up, and the fragments, with various initial speeds, start to move away from the centre of mass G which may be regarded as a fixed point. In the subsequent motion the acceleration of each fragment is directed towards G. Moreover, in accordance with the laws of physics of the Z–universe, there are positive constants k_1, k_2 and R such that when a fragment is at a distance x from G, the magnitude of its acceleration is $k_1 x^3$ if $x < R$ and is $k_2 x^{-4}$ if $x \geqslant R$. The initial speed of a fragment is denoted by u.

(i) For $x < R$, write down a differential equation for the speed v, and hence determine v in terms of u, k_1 and x for $x < R$.

(ii) Show that if $u < a$, where $2a^2 = k_1 R^4$, then the fragment does not reach a distance R from G.

(iii) Show that if $u \geqslant b$, where $6b^2 = 3k_1 R^4 + 4k_2/R^3$, then from the moment of the explosion the fragment is always moving away from G.

(iv) If $a < u < b$, determine in terms of k_2, b and u the maximum distance from G attained by the fragment.

STEP III 2009 Question 11 (Mechanics)

11 A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude Mf acting in the direction of its motion. When it entered the cloud, the comet had mass M and speed V. After a time t, it has travelled a distance x through the cloud, its mass is $M(1 + bx)$, where b is a positive constant, and its speed is v.

(i) In the case when $f = 0$, write down an equation relating V, x, v and b. Hence find an expression for x in terms of b, V and t.

(ii) In the case when f is a non-zero constant, use Newton's second law in the form

force = rate of change of momentum

to show that

$$v = \frac{ft + V}{1 + bx}\,.$$

Hence find an expression for x in terms of b, V, f and t.

Show that it is possible, if b, V and f are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.

STEP III 2011 Question 1 (Pure)

1 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right)u = 0\,.$$

(ii) Show that substituting $y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

STEP II 2008 Question 7 (Pure)

7 **(i)** By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1} y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

STEP I 2008 Question 8 (Pure)

8 **(i)** The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0\,. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

(ii) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0\,.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

STEP III 1996 Question 14 (Probability and Statistics)

14 Whenever I go cycling I start with my bike in good working order. However if all is well at time t, the probability that I get a puncture in the small interval $(t, t + \delta t)$ is $\alpha\,\delta t$. How many punctures can I expect to get on a journey during which my total cycling time is T?

When I get a puncture I stop immediately to repair it and the probability that, if I am repairing it at time t, the repair will be completed in time $(t, t + \delta t)$ is $\beta\,\delta t$. If $p(t)$ is the probability that I am repairing a puncture at time t, write down an equation relating $p(t)$ to $p(t + \delta t)$, and derive from this a differential equation relating $p'(t)$ and $p(t)$. Show that

$$p(t) = \frac{\alpha}{\alpha + \beta}(1 - \mathrm{e}^{-(\alpha+\beta)t})$$

satisfies this differential equation with the appropriate initial condition.

Find an expression, involving α, β and T, for the time expected to be spent mending punctures during a journey of total time T. Hence, or otherwise, show that, the fraction of the journey expected to be spent mending punctures is given approximately by

$$\frac{\alpha T}{2} \quad \text{if } (\alpha + \beta)T \text{ is small,}$$

and by

$$\frac{\alpha}{\alpha + \beta} \quad \text{if } (\alpha + \beta)T \text{ is large.}$$