STEP Past Papers by Topic

STEP Topic – Differential equations

STEP II 1992 Question 2 (Pure)

2 Suppose that y satisfies the differential equation

$$y = x\frac{\mathrm{d}y}{\mathrm{d}x} - \cosh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right). \qquad (*)$$

By differentiating both sides of $(*)$ with respect to x, show that either

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \qquad \text{or} \qquad x - \sinh\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

Find the general solutions of each of these two equations. Determine the solutions of $(*)$.

STEP II 1990 Question 7 (Pure)

7 A damped system with feedback is modelled by the equation

$$f'(t) + f(t) - kf(t-1) = 0, \qquad (\dagger)$$

where k is a given non-zero constant. Show that (non-zero) solutions for f of the form $f(t) = Ae^{pt}$, where A and p are constants, are possible provided p satisfies

$$p + 1 = ke^{-p}. \qquad (*)$$

Show also, by means of a sketch, or otherwise, that equation $(*)$ can have $0, 1$ or 2 real roots, depending on the value of k, and find the set of values of k for which such solutions of $(\dagger)$ exist. For what set of values of k do such solutions tend to zero as $t \to +\infty$?

STEP II 1991 Question 8 (Pure)

8 Solve the quadratic equation $u^2 + 2u\sinh x - 1 = 0$, giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\sinh x - 1 = 0$$

which satisfies $y = 0$ and $y' > 0$ at $x = 0$.

Find the solution of the differential equation

$$\sinh x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2\frac{\mathrm{d}y}{\mathrm{d}x} - \sinh x = 0$$

which satisfies $y = 0$ at $x = 0$.

STEP II 1987 Question 6 (Pure)

6 The functions $x(t)$ and $y(t)$ satisfy the simultaneous differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 2x - 5y = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} + ax - 2y = 2\cos t,$$

subject to $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ at $t = 0$.

Solve these equations for x and y in the case when $a = 1$.

Without solving the equations explicitly, state briefly how the form of the solutions for x and y if $a > 1$ would differ from the form when $a = 1$.

STEP I 1990 Question 7 (Pure)

7 Let y, u, v, P and Q all be functions of x. Show that the substitution $y = uv$ in the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$$

leads to an equation for $\frac{\mathrm{d}v}{\mathrm{d}x}$ in terms of x, Q and u, provided that u satisfies a suitable first order differential equation.

Hence or otherwise solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x+1} = (x+1)^{\frac{5}{2}},$$

given that $y(1) = 0$. For what set of values of x is the solution valid?

STEP III 1991 Question 5 (Pure)

5 The curve C has the differential equation in polar coordinates

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} + 4r = 5\sin 3\theta, \qquad \text{for} \quad \frac{\pi}{5} \leqslant \theta \leqslant \frac{3\pi}{5},$$

and, when $\theta = \frac{\pi}{2}$, $r = 1$ and $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2$.

Show that C forms a closed loop and that the area of the region enclosed by C is

$$\frac{\pi}{5} + \frac{25}{48}\left[\sin\left(\frac{\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\right].$$

STEP II 2004 Question 8 (Pure)

8 Let x satisfy the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(1 - x^n\right)^{1/n}$$

and the condition $x = 0$ when $t = 0$.

(i) Solve the equation in the case $n = 1$ and sketch the graph of the solution for $t > 0$.

(ii) Prove that $1 - x < (1 - x^2)^{1/2}$ for $0 < x < 1$.

Use this result to sketch the graph of the solution in the case $n = 2$ for $0 < t < \frac{1}{2}\pi$, using the same axes as your previous sketch.

By setting $x = \sin y$, solve the equation in this case.

(iii) Use the result (which you need not prove)

$$(1 - x^2)^{1/2} < (1 - x^3)^{1/3} \quad \text{for} \quad 0 < x < 1\,,$$

to sketch, without solving the equation, the graph of the solution of the equation in the case $n = 3$ using the same axes as your previous sketches. Use your sketch to show that $x = 1$ at a value of t less than $\frac{1}{2}\pi$.

STEP II 2005 Question 8 (Pure)

8 For $x \geqslant 0$ the curve C is defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 y^2}{(1+x^2)^{5/2}}$$

with $y = 1$ when $x = 0$. Show that

$$\frac{1}{y} = \frac{2+3x^2}{3(1+x^2)^{3/2}} + \frac{1}{3}$$

and hence that for large positive x

$$y \approx 3 - \frac{9}{x}\,.$$

Draw a sketch of C.

On a separate diagram draw a sketch of the two curves defined for $x \geqslant 0$ by

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{x^3 z^3}{2(1+x^2)^{5/2}}$$

with $z = 1$ at $x = 0$ on one curve, and $z = -1$ at $x = 0$ on the other.

STEP II 2003 Question 8 (Pure)

8 It is given that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} + k\left(\frac{t^2 - 3t + 2}{t+1}\right) y = 0\,,$$

where k is a constant, and $y = A$ when $t = 0$, where A is a positive constant. Find y in terms of t, k and A.

Show that y has two stationary values whose ratio is $(3/2)^{6k}\mathrm{e}^{-5k/2}$.

Describe the behaviour of y as $t \to +\infty$ for the case where $k > 0$ and for the case where $k < 0$.

In separate diagrams, sketch the graph of y for $t > 0$ for each of these cases.

STEP I 2001 Question 8 (Pure)

8 Given that $y = x$ and $y = 1 - x^2$ satisfy the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \mathrm{p}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{q}(x)y = 0\ , \qquad (*)$$

show that $\mathrm{p}(x) = -2x(1+x^2)^{-1}$ and $\mathrm{q}(x) = 2(1+x^2)^{-1}$.

Show also that $ax + b(1-x^2)$ satisfies the differential equation for any constants a and b.

Given instead that $y = \cos^2(\frac{1}{2}x^2)$ and $y = \sin^2(\frac{1}{2}x^2)$ satisfy the equation $(*)$, find $\mathrm{p}(x)$ and $\mathrm{q}(x)$.

STEP I 2003 Question 8 (Pure)

8 A liquid of fixed volume V is made up of two chemicals A and B. A reaction takes place in which A converts to B. The volume of A at time t is xV and the volume of B at time t is yV where x and y depend on t and $x + y = 1$. The rate at which A converts into B is given by $kVxy$, where k is a positive constant. Show that if both x and y are strictly positive at the start, then at time t

$$y = \frac{D\mathrm{e}^{kt}}{1 + D\mathrm{e}^{kt}},$$

where D is a constant.

Does A ever completely convert to B? Justify your answer.

STEP II 1995 Question 8 (Pure)

8 If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

$$x(t) = K + K\frac{1 - \mathrm{e}^{-2Kt}}{1 + \mathrm{e}^{-2Kt}}.$$

Sketch, for $t \geqslant 0$, the graph of x against t. What happens to $x(t)$ as $t \to \infty$?

Now suppose that the situation is as described in the first paragraph, except that we remove the bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x. By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \to K + \alpha$ as $t \to \infty$.

STEP III 2007 Question 5 (Pure)

5 Let $y = \ln(x^2 - 1)$, where $x > 1$, and let r and θ be functions of x determined by $r = \sqrt{x^2 - 1}$ and $\coth\theta = x$. Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cosh\theta}{r} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\cosh 2\theta}{r^2},$$

and find an expression in terms of r and θ for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Find, with proof, a similar formula for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ in terms of r and θ.

STEP III 2015 Question 8 (Pure)

8 **(i)** Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y + x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0\,.$$

Sketch a solution in the x-y plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2 + y^2)\big)\,\frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A.

STEP II 2018 Question 8 (Pure)

8 **(i)** Use the substitution $v = \sqrt{y}$ to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{1}{2}} - \beta y \qquad (y \geqslant 0, \quad t \geqslant 0),$$

where α and β are positive constants. Find the non-constant solution $y_1(x)$ that satisfies $y_1(0) = 0$.

(ii) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y^{\frac{2}{3}} - \beta y \qquad (y \geqslant 0, \quad t \geqslant 0),$$

where α and β are positive constants. Find the non-constant solution $y_2(x)$ that satisfies $y_2(0) = 0$.

(iii) In the case $\alpha = \beta$, sketch $y_1(x)$ and $y_2(x)$ on the same axes, indicating clearly which is $y_1(x)$ and which is $y_2(x)$. You should explain how you determined the positions of the curves relative to each other.

STEP III 2005 Question 2 (Pure)

2 Find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{xy}{x^2+a^2}$, where $a \neq 0$, and show that it can be written in the form $y^2(x^2+a^2) = c^2$, where c is an arbitrary constant. Sketch this curve.

Find an expression for $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^2+y^2)$ and show that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2+y^2) = 2\left(1 - \frac{c^2}{(x^2+a^2)^2}\right) + \frac{8c^2x^2}{(x^2+a^2)^3} .$$

(i) Show that, if $0 < c < a^2$, the points on the curve whose distance from the origin is least are $\left(0\,,\ \pm\dfrac{c}{a}\right)$.

(ii) If $c > a^2$, determine the points on the curve whose distance from the origin is least.

STEP III 1994 Question 4 (Pure)

4 Find the two solutions of the differential equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y$$

which pass through the point (a, b^2), where $b \neq 0$.

Find two distinct points $(a_1, 1)$ and $(a_2, 1)$ such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1, 1)$ and $(a_2, 1)$.

Now sketch sufficient members of the family of solutions (for varying a and b) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

STEP III 2007 Question 8 (Pure)

8 **(i)** Find functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$ such that $u = x$ and $u = \mathrm{e}^{-x}$ both satisfy the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \mathrm{a}(x)\frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{b}(x)u = 0\,.$$

For these functions $\mathrm{a}(x)$ and $\mathrm{b}(x)$, write down the general solution of the equation.

Show that the substitution $y = \dfrac{1}{3u}\dfrac{\mathrm{d}u}{\mathrm{d}x}$ transforms the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2 + \frac{x}{1+x}y = \frac{1}{3(1+x)} \qquad (*)$$

into

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{x}{1+x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{1+x}u = 0$$

and hence show that the solution of equation $(*)$ that satisfies $y = 0$ at $x = 0$ is given by $y = \dfrac{1-\mathrm{e}^{-x}}{3(x+\mathrm{e}^{-x})}$.

(ii) Find the solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{x}{1-x}y = \frac{1}{1-x}$$

that satisfies $y = 2$ at $x = 0$.

STEP II 2014 Question 5 (Pure)

5 Given that $y = xu$, where u is a function of x, write down an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$.

Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

STEP I 2010 Question 6 (Pure)

6 Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

STEP I 2013 Question 7 (Pure)

7 **(i)** Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

STEP III 2013 Question 7 (Pure)

7 **(i)** Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+y^3=0$ with $y=1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ at $x=0$, and let

$$\mathrm{E}(x)=\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)|\leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2}+x\dfrac{\mathrm{d}v}{\mathrm{d}x}+\sinh v=0$ with $v=\ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}=0$ at $x=0$, and let

$$\mathrm{E}(x)=\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2+2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x}\leqslant 0$ for $x\geqslant 0$ and deduce that $\cosh v(x)\leqslant \frac{5}{3}$ for $x\geqslant 0$.

(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+(5\cosh x-4\sinh x-3)\frac{\mathrm{d}w}{\mathrm{d}x}+(w\cosh w+2\sinh w)=0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x}=\dfrac{1}{\sqrt{2}}$ and $w=0$ at $x=0$. Show that $\cosh w(x)\leqslant\frac{5}{4}$ for $x\geqslant 0$.

STEP I 1995 Question 6 (Pure)

6 **(i)** In the differential equation

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y} = \mathrm{e}^{2x}$$

make the substitution $u = 1/y$, and hence show that the general solution of the original equation is

$$y = \frac{1}{A\mathrm{e}^{x} - \mathrm{e}^{2x}}\,.$$

(ii) Use a similar method to solve the equation

$$\frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{y^2} = \mathrm{e}^{2x}.$$
