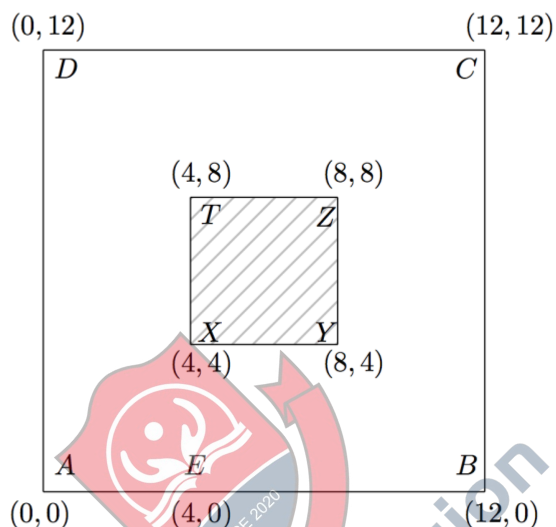
STEP Past Papers by Topic

STEP Topic – Differentiation

STEP I 1991 Question 4 (Pure)

4



The above diagram is a plan of a prison compound. The outer square $ABCD$ represents the walls of the compound (whose height may be neglected), while the inner square $XYZT$ is the Black Tower, a solid stone structure. A guard patrols along segment AE of the walls, for a distance of up to 4 units from A. Determine the distance from A of points at which the area of the courtyard that he can see is

(i) as small as possible,

(ii) as large as possible.

[**Hint.** It is suggested that you express the area he *cannot* see in terms of p, his distance from A.]

STEP I 1991 Question 2 (Pure)

2 Frosty the snowman is made from two uniform spherical snowballs, of initial radii $2R$ and $3R$. The smaller (which is his head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for both snowballs. During melting each snowball remains spherical and uniform. When Frosty is half his initial height, find the ratio of his volume to his initial volume.

If V and S denote his total volume and surface area respectively, find the maximum value of $\dfrac{\mathrm{d}V}{\mathrm{d}S}$ up to the moment when his head disappears.

STEP I 1988 Question 2 (Pure)

2 The function f and g are related (for all real x) by

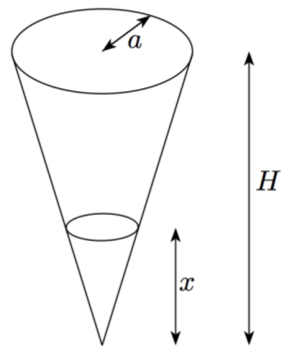
$$\mathrm{g}(x) = \mathrm{f}(x) + \frac{1}{\mathrm{f}(x)}\,.$$

Express $\mathrm{g}'(x)$ and $\mathrm{g}''(x)$ in terms of $\mathrm{f}(x)$ and its derivatives.

If $\mathrm{f}(x) = 4 + \cos 2x + 2\sin x$, find the stationary points of g for $0 \leqslant x \leqslant 2\pi$, and determine which are maxima and which are minima.

STEP I 1992 Question 9 (Pure)

9 The diagram shows a coffee filter consisting of an inverted hollow right circular cone of height H cm and base radius a cm.



When the water level is x cm above the vertex, water leaves the cone at a rate $Ax\ \mathrm{cm^3 sec^{-1}}$, where A is a positive constant. Suppose that the cone is initially filled to a height h cm with $0 < h < H$. Show that it will take $\pi a^2 h^2/(2AH^2)$ seconds to empty.

Suppose now that the cone is initially filled to a height h cm, but that water is poured in at a constant rate $B\ \mathrm{cm^3 sec^{-1}}$ and continues to drain as before. Establish, by considering the sign of $\mathrm{d}x/\mathrm{d}t$, or otherwise, what will happen subsequently to the water level in the different cases that arise. (You are not asked to find an explicit formula for x.)

STEP I 1996 Question 1 (Pure)

1 A cylindrical biscuit tin has volume V and surface area S (including the ends). Show that the minimum possible surface area for a given value of V is $S = 3(2\pi V^2)^{1/3}$. For this value of S show that the volume of the largest sphere which can fit inside the tin is $\frac{2}{3}V$, and find the volume of the smallest sphere into which the tin fits.

STEP I 1997 Question 2 (Pure)

2 **(i)** If

$$f(x) = \tan^{-1} x + \tan^{-1}\left(\frac{1-x}{1+x}\right),$$

find $f'(x)$. Hence, or otherwise, find a simple expression for $f(x)$.

(ii) Suppose that y is a function of x with $0 < y < (\pi/2)^{1/2}$ and

$$x = y \sin y^2$$

for $0 < x < (\pi/2)^{1/2}$. Show that (for this range of x)

$$\frac{dy}{dx} = \frac{y}{x + 2y^2\sqrt{y^2 - x^2}}.$$

STEP II 2008 Question 4 (Pure)

4 A curve is given by

$$x^2 + y^2 + 2axy = 1,$$

where a is a constant satisfying $0 < a < 1$. Show that the gradient of the curve at the point P with coordinates (x, y) is

$$-\frac{x + ay}{ax + y},$$

provided $ax + y \neq 0$. Show that θ, the acute angle between OP and the normal to the curve at P, satisfies

$$\tan\theta = a|y^2 - x^2|\,.$$

Show further that, if $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$ at P, then:

(i) $a(x^2 + y^2) + 2xy = 0\,;$

(ii) $(1 + a)(x^2 + y^2 + 2xy) = 1\,;$

(iii) $\tan\theta = \dfrac{a}{\sqrt{1 - a^2}}\,.$

STEP I 2000 Question 7 (Pure)

7 Let

$$f(x) = ax - \frac{x^3}{1+x^2},$$

where a is a constant. Show that, if $a \geqslant 9/8$, then $f'(x) \geqslant 0$ for all x.

STEP I 2011 Question 1 (Pure)

1 **(i)** Show that the gradient of the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $b\neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p,q) lies on both the straight line $ax+by=1$ and the curve $\dfrac{a}{x}+\dfrac{b}{y}=1$, where $ab\neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p=\pm q$.

Show further that either $(a-b)^2=1$ or $(a+b)^2=1$.

(ii) Show that if the straight line $ax+by=1$, where $ab\neq 0$, is a normal to the curve $\dfrac{a}{x}-\dfrac{b}{y}=1$, then $a^2-b^2=\frac{1}{2}$.

STEP III 2015 Question 7 (Pure)

7 An operator D is defined, for any function f, by

$$\mathrm{Df}(x) = x\frac{\mathrm{df}(x)}{\mathrm{d}x}.$$

The notation D^n means that D is applied n times; for example

$$\mathrm{D}^2\mathrm{f}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{df}(x)}{\mathrm{d}x}\right).$$

Show that, for any constant a, $\mathrm{D}^2x^a = a^2x^a$.

(i) Show that if $\mathrm{P}(x)$ is a polynomial of degree r (where $r \geqslant 1$) then, for any positive integer n, $\mathrm{D}^n\mathrm{P}(x)$ is also a polynomial of degree r.

(ii) Show that if n and m are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0.$$

STEP I 2015 Question 7 (Pure)

7 Let

$$f(x) = 3ax^2 - 6x^3$$

and, for each real number a, let $M(a)$ be the greatest value of $f(x)$ in the interval $-\frac{1}{3} \leqslant x \leqslant 1$. Determine $M(a)$ for $a \geqslant 0$. [The formula for $M(a)$ is different in different ranges of a; you will need to identify three ranges.]

STEP I 1994 Question 2 (Pure)

2 Given that a is constant, differentiate the following expressions with respect to x:

(i) x^a;

(ii) a^x;

(iii) x^x;

(iv) $x^{(x^x)}$;

(v) $(x^x)^x$.

STEP I 2008 Question 2 (Pure)

2 The variables t and x are related by $t = x + \sqrt{x^2 + 2bx + c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t - x}{t + b} \, ,$$

and hence integrate $\dfrac{1}{\sqrt{x^2 + 2bx + c}}$.

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x + b > 0$ but that your result does not hold in the case $b^2 = c$ if $x + b < 0$.
