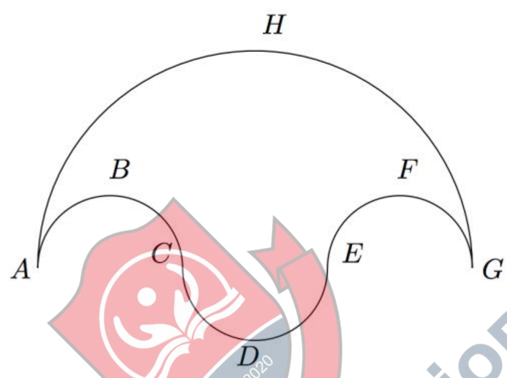
STEP Past Papers by Topic

STEP Topic – Euclidean geometry

STEP I 1989 Question 1 (Pure)

1

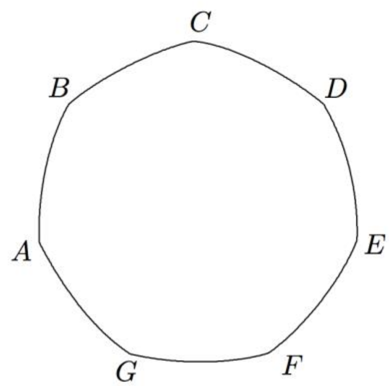


In the above diagram, ABC, CDE, EFG and AHG are semicircles and A, C, E, G lie on a straight line. The radii of ABC, EFG, AHG are h, h and r respectively, where $2h < r$. Show that the area enclosed by $ABCDEFGH$ is equal to that of a circle with diameter HD.

Each semicircle is now replaced by a portion of a parabola, with vertex at the midpoint of the semicircle arc, passing through the endpoints (so, for example, ABC is replaced by part of a parabola passing through A and C and with vertex at B). Find a formula in terms of r and h for the area enclosed by $ABCDEFGH$.

STEP I Specimen Question 6 (Pure)

6



The diagram shows a cross-section, parallel to its faces, of a British 50 pence coin. The seven arcs $AB, BC, \ldots, FG, GA$ are all of equal length and each arc is formed from the circle having its centre at the vertex diametrically opposite the mid-point of the arc. Given that the radius of each of these circles is a, show that the area of a face of the coin is

$$\frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right).$$

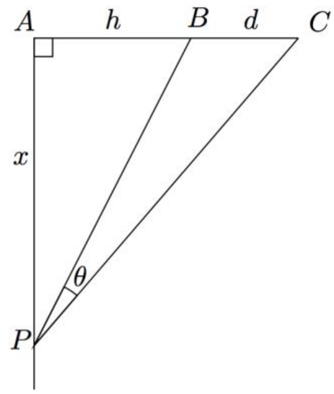
STEP I 1989 Question 4 (Pure)

4 Six points A, B, C, D, E and F lie in three dimensional space and are in general positions, that is, no three are collinear and no four lie on a plane. All possible line segments joining pairs of points are drawn and coloured either gold or silver. Prove that there is a triangle whose edges are entirely of one colour. [**Hint**: consider segments radiating from A.]

Give a sketch showing that the result is false for five points in general positions.

STEP I Specimen Question 2 (Pure)

2 In the figure, the angle PAC is a right angle. $AB = h, BC = d, AP = x$ and the angle BPC is θ. Express $\tan\theta$ in terms of h, d and x.



Given that x may be varied, deduce, or find otherwise, the value of x (in terms of the constants d and h) for which θ has its maximum value.

STEP I 1990 Question 1 (Pure)

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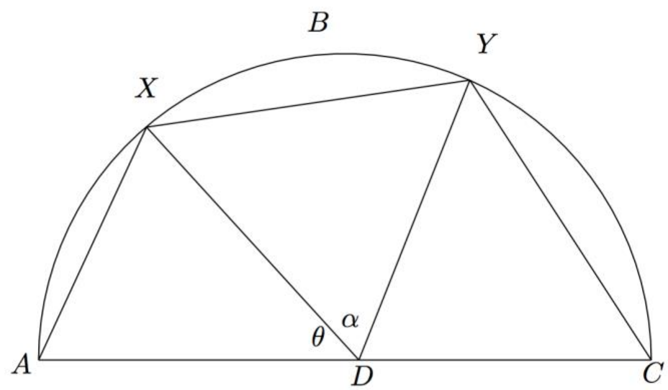


In the above diagram, $ABCD$ represents a semicircular window of fixed radius r and centre D, and $AXYC$ is a quadrilateral blind. If $\angle XDY = \alpha$ is fixed and $\angle ADX = \theta$ is variable, determine the value of θ which gives the blind **maximum** area.

If now α is allowed to vary but r remains fixed, find the maximum possible area of the blind.

STEP I 1988 Question 8 (Pure)

8 $ABCD$ is a skew (non-planar) quadrilateral, and its pairs of opposite sides are equal, i.e. $AB = CD$ and $BC = AD$. Prove that the line joining the midpoints of the diagonals AC and BD is perpendicular to each diagonal.

STEP III 1992 Question 7 (Pure)

7 The points P and R lie on the sides AB and AD, respectively, of the parallelogram $ABCD$. The point Q is the fourth vertex of the parallelogram $APQR$. Prove that BR, CQ and DP meet in a point.

STEP III 2004 Question 4 (Pure)

4 The triangle OAB is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle C_0 has its centre at the midpoint of the base AB of the triangle, and the sides OA and OB of the triangle are both tangent to the semi-circle. $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \ldots$ are circles such that C_n is tangent to C_{n-1} and to sides OA and OB of the triangle.

Let r_n be the radius of C_n. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1 - \sin\alpha}{1 + \sin\alpha}\,.$$

Let S be the total area of the semi-circle C_0 and the circles C_1, C_2, C_3, $\ldots$. Show that

$$S = \frac{1 + \sin^2\alpha}{4\sin\alpha}\,\pi r_0^2\,.$$

Show that there are values of α for which S is more than four fifths of the area of triangle OAB.

STEP II 2005 Question 5 (Pure)

5 The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1) \,,$$

where $q = \dfrac{b+c}{a}$. Deduce that

$$R \leqslant \frac{1}{\pi(\pi - 1)} \,.$$

STEP I 2000 Question 5 (Pure)

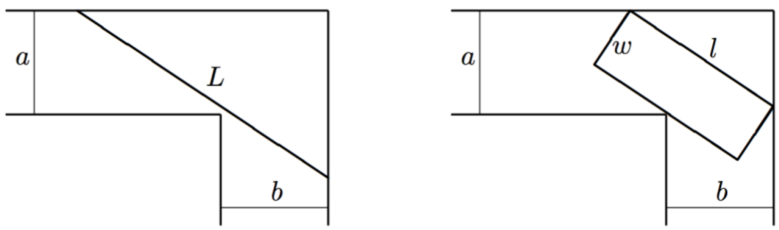
5 Arthur and Bertha stand at a point O on an inclined plane. The steepest line in the plane through O makes an angle θ with the horizontal. Arthur walks uphill at a steady pace in a straight line which makes an angle α with the steepest line. Bertha walks uphill at the same speed in a straight line which makes an angle β with the steepest line (and is on the same side of the steepest line as Arthur). Show that, when Arthur has walked a distance d, the distance between Arthur and Bertha is $2d|\sin\frac{1}{2}(\alpha-\beta)|$. Show also that, if $\alpha \neq \beta$, the line joining Arthur and Bertha makes an angle ϕ with the vertical, where

$$\cos\phi = \sin\theta\sin\tfrac{1}{2}(\alpha+\beta).$$

STEP II 2004 Question 4 (Pure)

4

Figure 1 Figure 2



(i) An attempt is made to move a rod of length L from a corridor of width a into a corridor of width b, where $a \neq b$. The corridors meet at right angles, as shown in Figure 1 and the rod remains horizontal. Show that if the attempt is to be successful then

$$L \leqslant a \operatorname{cosec} \alpha + b \sec \alpha \ ,$$

where α satisfies

$$\tan^3 \alpha = \frac{a}{b} \ .$$

(ii) An attempt is made to move a rectangular table-top, of width w and length l, from one corridor to the other, as shown in the Figure 2. The table-top remains horizontal. Show that if the attempt is to be successful then

$$l \leqslant a \operatorname{cosec} \beta + b \sec \beta - 2w \operatorname{cosec} 2\beta,$$

where β satisfies

$$w = \left(\frac{a - b\tan^3 \beta}{1 - \tan^2 \beta} \right) \cos \beta \ .$$

STEP II 1996 Question 7 (Pure)

7 Consider a fixed square $ABCD$ and a variable point P in the plane of the square. We write the perpendicular distance from P to AB as p, from P to BC as q, from P to CD as r and from P to DA as s. (Remember that distance is never negative, so $p, q, r, s \geqslant 0$.) If $pr = qs$, show that the only possible positions of P lie on two straight lines and a circle and that every point on these two lines and a circle is indeed a possible position of P.

STEP III 1994 Question 3 (Pure)

3 Describe geometrically the possible intersections of a plane with a sphere.

Let P_1 and P_2 be the planes with equations

$$3x - y - 1 = 0,$$
$$x - y + 1 = 0,$$

respectively, and let S_1 and S_2 be the spheres with equations

$$x^2 + y^2 + z^2 = 7,$$
$$x^2 + y^2 + z^2 - 6y - 4z + 10 = 0,$$

respectively. Let C_1 be the intersection of P_1 and S_1, let C_2 be the intersection of P_2 and S_2 and let L be the intersection of P_1 and P_2. Find the points where L meets each of S_1 and S_2. Determine, giving your reasons, whether the circles C_1 and C_2 are linked.

STEP II 2012 Question 6 (Pure)

6 A cyclic quadrilateral $ABCD$ has sides AB, BC, CD and DA of lengths a, b, c and d, respectively. The area of the quadrilateral is Q, and angle DAB is θ.

Find an expression for $\cos\theta$ in terms of a, b, c and d, and an expression for $\sin\theta$ in terms of a, b, c, d and Q. Hence show that

$$16Q^2 = 4(ad+bc)^2 - (a^2+d^2-b^2-c^2)^2\,,$$

and deduce that

$$Q^2 = (s-a)(s-b)(s-c)(s-d)\,,$$

where $s = \frac{1}{2}(a+b+c+d)$.

Deduce a formula for the area of a triangle with sides of length a, b and c.

STEP II 2003 Question 4 (Pure)

4 The line $y = d$, where $d > 0$, intersects the circle $x^2 + y^2 = R^2$ at G and H. Show that the area of the minor segment GH is equal to

$$R^2 \arccos\left(\frac{d}{R}\right) - d\sqrt{R^2 - d^2}\,. \qquad (*)$$

In the following cases, the given line intersects the given circle. Determine how, in each case, the expression $(*)$ should be modified to give the area of the minor segment.

(i) Line: $y = c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.

(ii) Line: $y = mx + c$; circle: $x^2 + y^2 = R^2$.

(iii) Line: $y = mx + c$; circle: $(x - a)^2 + (y - b)^2 = R^2$.

STEP II 2001 Question 3 (Pure)

3 The cuboid $ABCDEFGH$ is such AE, BF, CG, DH are perpendicular to the opposite faces $ABCD$ and $EFGH$, and $AB = 2, BC = 1, AE = \lambda$. Show that if α is the acute angle between the diagonals AG and BH then

$$\cos\alpha = \left|\frac{3-\lambda^2}{5+\lambda^2}\right|$$

Let R be the ratio of the volume of the cuboid to its surface area. Show that $R < \frac{1}{3}$ for all possible values of λ.

Prove that, if $R \geqslant \frac{1}{4}$, then $\alpha \leqslant \arccos\frac{1}{9}$.

STEP I 1997 Question 5 (Pure)

5 Four rigid rods AB, BC, CD and DA are freely jointed together to form a quadrilateral in the plane. Show that if P, Q, R, S are the mid-points of the sides AB, BC, CD, DA, respectively, then

$$|AB|^2 + |CD|^2 + 2|PR|^2 = |AD|^2 + |BC|^2 + 2|QS|^2.$$

Deduce that $|PR|^2 - |QS|^2$ remains constant however the vertices move. (Here $|PR|$ denotes the length of PR.)

STEP I 2002 Question 6 (Pure)

6 A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three sides of the pyramid are of length b and its volume is V. Given that the formula for the volume of any pyramid is $\frac{1}{3} \times$ area of base $\times$ height, show that

$$V = \frac{1}{12}a^2(3b^2 - a^2)^{\frac{1}{2}} \ .$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Show that the new height, h, of the pyramid is given by

$$h^2 = \frac{a^2(3b^2 - a^2)}{4b^2 - a^2} \ .$$

Find, in terms of a and b, the angle between the equilateral triangle and the horizontal.

STEP I 2001 Question 1 (Pure)

1 The points A, B and C lie on the sides of a square of side 1 cm and no two points lie on the same side. Show that the length of at least one side of the triangle ABC must be less than or equal to $(\sqrt{6} - \sqrt{2})$ cm.

STEP II 2002 Question 6 (Pure)

6 The lines l_1, l_2 and l_3 lie in an inclined plane P and pass through a common point A. The line l_2 is a line of greatest slope in P. The line l_1 is perpendicular to l_3 and makes an acute angle α with l_2. The angles between the horizontal and l_1, l_2 and l_3 are $\pi/6$, β and $\pi/4$, respectively. Show that $\cos\alpha \sin\beta = \frac{1}{2}$ and find the value of $\sin\alpha\sin\beta$. Deduce that $\beta = \pi/3$.

The lines l_1 and l_3 are rotated in P about A so that l_1 and l_3 remain perpendicular to each other. The new acute angle between l_1 and l_2 is θ. The new angles which l_1 and l_3 make with the horizontal are ϕ and 2ϕ, respectively. Show that

$$\tan^2\theta = \frac{3+\sqrt{13}}{2}\,.$$

STEP I 2012 Question 6 (Pure)

6 A thin circular path with diameter AB is laid on horizontal ground. A vertical flagpole is erected with its base at a point D on the diameter AB. The angles of elevation of the top of the flagpole from A and B are α and β respectively (both are acute). The point C lies on the circular path with DC perpendicular to AB and the angle of elevation of the top of the flagpole from C is ϕ. Show that $\cot\alpha\cot\beta = \cot^2\phi$.

Show that, for any p and q,

$$\cos p\cos q\sin^2\tfrac{1}{2}(p+q) - \sin p\sin q\cos^2\tfrac{1}{2}(p+q) = \tfrac{1}{2}\cos(p+q) - \tfrac{1}{2}\cos(p+q)\cos(p-q).$$

Deduce that, if p and q are positive and $p+q \leqslant \tfrac{1}{2}\pi$, then

$$\cot p\cot q \geqslant \cot^2\tfrac{1}{2}(p+q)$$

and hence show that $\phi \leqslant \tfrac{1}{2}(\alpha+\beta)$ when $\alpha+\beta \leqslant \tfrac{1}{2}\pi$.

STEP I 2015 Question 4 (Pure)

4 The midpoint of a rod of length $2b$ slides on the curve $y = \frac{1}{4}x^2$, $x \geqslant 0$, in such a way that the rod is always tangent, at its midpoint, to the curve. Show that the curve traced out by one end of the rod can be written in the form

$$x = 2\tan\theta - b\cos\theta$$
$$y = \tan^2\theta - b\sin\theta$$

for some suitably chosen angle θ which satisfies $0 \leqslant \theta < \frac{1}{2}\pi$.

When one end of the rod is at a point A on the y-axis, the midpoint is at point P and $\theta = \alpha$. Let R be the region bounded by the following:

the curve $y = \frac{1}{4}x^2$ between the origin and P;

the y-axis between A and the origin;

the half-rod AP.

Show that the area of R is $\frac{2}{3}\tan^3\alpha$.

STEP II 2014 Question 1 (Pure)

1 In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leqslant \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$\big(1 + 2\cos(\alpha+\beta)\big)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0\,. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leqslant \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45°$ and (b) $\alpha = 30°$, $\beta = 90°$, and illustrate each case with a sketch.

STEP I 2015 Question 6 (Pure)

6 The vertices of a plane quadrilateral are labelled A, B, A' and B', in clockwise order. A point O lies in the same plane and within the quadrilateral. The angles AOB and $A'OB'$ are right angles, and $OA = OB$ and $OA' = OB'$.

Use position vectors relative to O to show that the midpoints of AB, BA', $A'B'$ and $B'A$ are the vertices of a square.

Given that the lengths of OA and OA' are fixed (and the conditions of the first paragraph still hold), find the value of angle BOA' for which the area of the square is greatest.

STEP II 2015 Question 2 (Pure)

2 In the triangle ABC, angle $BAC = \alpha$ and angle $CBA = 2\alpha$, where 2α is acute, and $BC = x$. Show that $AB = (3 - 4\sin^2\alpha)x$.

The point D is the midpoint of AB and the point E is the foot of the perpendicular from C to AB. Find an expression for DE in terms of x.

The point F lies on the perpendicular bisector of AB and is a distance x from C. The points F and B lie on the same side of the line through A and C. Show that the line FC trisects the angle ACB.

STEP I 2009 Question 4 (Pure)

4 The sides of a triangle have lengths $p - q$, p and $p + q$, where $p > q > 0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta\,.$$

In the case $\alpha = 2\beta$, show that $\cos\beta = \frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

STEP I 2009 Question 8 (Pure)

8 **(i)** The equation of the circle C is

$$(x-2t)^2+(y-t)^2=t^2,$$

where t is a positive number. Show that C touches the line $y=0$.

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y=4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y=0$, $3y=4x$ and $4y+3x=15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.
