STEP Past Papers by Topic

STEP Topic – Functions

STEP II 1991 Question 7 (Pure)

7 The function g satisfies, for all positive x and y,

$$\mathrm{g}(x) + \mathrm{g}(y) = \mathrm{g}(z), \qquad (*)$$

where $z = xy/(x+y+1)$. By treating y as a constant, show that

$$\mathrm{g}'(x) = \frac{y^2+y}{(x+y+1)^2}\mathrm{g}'(z) = \frac{z(z+1)}{x(x+1)}\mathrm{g}'(z),$$

and deduce that $2\mathrm{g}'(1) = (u^2+u)\mathrm{g}'(u)$ for all u satisfying $0 < u < 1$. Now by treating u as a variable, show that

$$\mathrm{g}(u) = A\ln\left(\frac{u}{u+1}\right) + B,$$

where A and B are constants. Verify that g satisfies $(*)$ for a suitable value of B. Can A be determined from $(*)$?

The function f satisfies, for all positive x and y,

$$\mathrm{f}(x) + \mathrm{f}(y) = \mathrm{f}(z)$$

where $z = xy$. Show that $\mathrm{f}(x) = C\ln x$ where C is a constant.

STEP II 1987 Question 5 (Pure)

5 If $y = \mathrm{f}(x)$, then the inverse of f (when it exists) can be obtained from *Lagrange's identity*. This identity, which you may use without proof, is

$$\mathrm{f}^{-1}(y) = y + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}} \left[y - \mathrm{f}\,(y)\right]^{n},$$

provided the series converges.

(i) Verify Lagrange's identity when $\mathrm{f}(x) = \alpha x$, $(0 < \alpha < 2)$.

(ii) Show that one root of the equation

$$\tfrac{1}{2} = x - \tfrac{1}{4}x^3$$

is

$$x = \sum_{n=0}^{\infty} \frac{(3n)!}{n!\,(2n+1)!2^{4n+1}}$$

(iii) Find a solution for x, as a series in λ, of the equation

$$x = \mathrm{e}^{\lambda x}.$$

[You may assume that the series in part **(ii)** converges, and that the series in part **(iii)** converges for suitable λ.]

STEP I 1992 Question 7 (Pure)

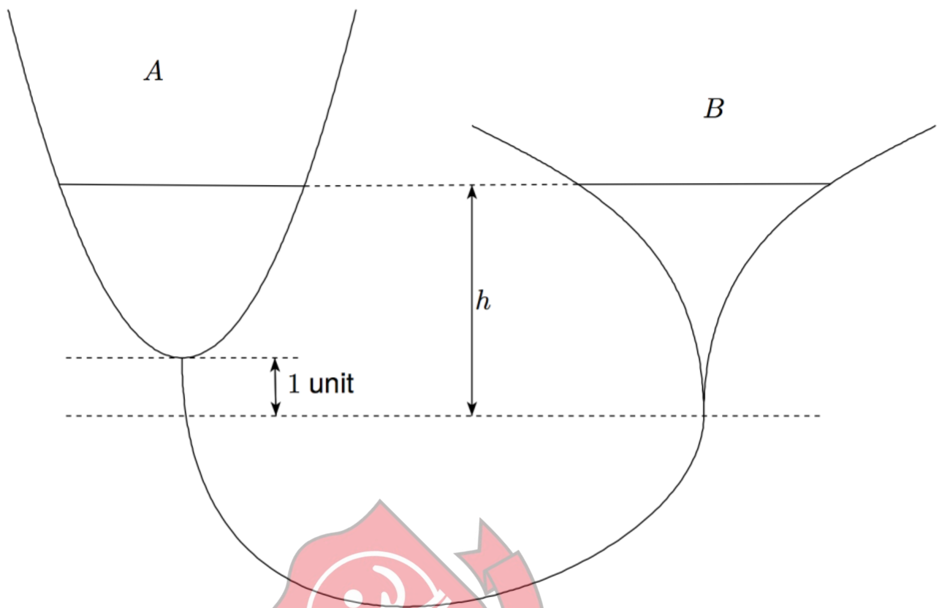
7 Let $\mathrm{g}(x) = ax + b$. Show that, if $\mathrm{g}(0)$ and $\mathrm{g}(1)$ are integers, then $\mathrm{g}(n)$ is an integer for all integers n.

Let $\mathrm{f}(x) = Ax^2 + Bx + C$. Show that, if $\mathrm{f}(-1), \mathrm{f}(0)$ and $\mathrm{f}(1)$ are integers, then $\mathrm{f}(n)$ is an integer for all integers n.

Show also that, if α is any real number and $\mathrm{f}(\alpha - 1)$, $\mathrm{f}(\alpha)$ and $\mathrm{f}(\alpha + 1)$ are integers, then $\mathrm{f}(\alpha + n)$ is an integer for all integers n.

STEP II 1987 Question 4 (Pure)

4



Two funnels A and B have surfaces formed by rotating the curves $y = x^2$ and $y = 2\sinh^{-1} x$ $(x > 0)$ above the y-axis. The bottom of B is one unit lower than the bottom of A and they are connected by a thin rubber tube with a tap in it. The tap is closed and A is filled with water to a depth of 4 units. The tap is then closed. When the water comes to rest, both surfaces are at a height h above the bottom of B, as shown in the diagram. Show that h satisfies the equation

$$h^2 - 3h + \sinh h = 15.$$

STEP II 1991 Question 3 (Pure)

3 It is given that x, y and z are distinct and non-zero, and that they satisfy

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Show that $x^2y^2z^2 = 1$ and that the value of $x + \dfrac{1}{y}$ is either $+1$ or -1.

STEP II 1994 Question 3 (Pure)

3 The function f satisfies $\mathrm{f}(0) = 1$ and

$$\mathrm{f}(x-y) = \mathrm{f}(x)\mathrm{f}(y) - \mathrm{f}(a-x)\mathrm{f}(a+y)$$

for some fixed number a and all x and y. Without making any further assumptions about the nature of the function show that $\mathrm{f}(a) = 0$.

Show that, for all t,

(i) $\mathrm{f}(t) = \mathrm{f}(-t)$,

(ii) $\mathrm{f}(2a) = -1$,

(iii) $\mathrm{f}(2a-t) = -\mathrm{f}(t)$,

(iv) $\mathrm{f}(4a+t) = \mathrm{f}(t)$.

Give an example of a non-constant function satisfying the conditions of the first paragraph with $a = \pi/2$. Give an example of an non-constant function satisfying the conditions of the first paragraph with $a = -2$.

STEP III 2001 Question 4 (Pure)

4 In this question, the function $\sin^{-1}$ is defined to have domain $-1 \leqslant x \leqslant 1$ and range $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$ and the function $\tan^{-1}$ is defined to have the real numbers as its domain and range $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

(i) Let

$$g(x) = \frac{2x}{1+x^2}, \qquad -\infty < x < \infty .$$

Sketch the graph of $g(x)$ and state the range of g.

(ii) Let

$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \qquad -\infty < x < \infty .$$

Show that $f(x) = 2\tan^{-1} x$ for $-1 \leqslant x \leqslant 1$ and $f(x) = \pi - 2\tan^{-1} x$ for $x \geqslant 1$.

Sketch the graph of $f(x)$.

STEP II 1998 Question 7 (Pure)

7

$$
\begin{aligned}
\mathrm{f}(x) &= \tan x - x, \\
\mathrm{g}(x) &= 2 - 2\cos x - x\sin x, \\
\mathrm{h}(x) &= 2x + x\cos 2x - \tfrac{3}{2}\sin 2x, \\
\mathrm{F}(x) &= \frac{x(\cos x)^{1/3}}{\sin x}.
\end{aligned}
$$

(i) By considering $\mathrm{f}(0)$ and $\mathrm{f}'(x)$, show that $\mathrm{f}(x) > 0$ for $0 < x < \frac{1}{2}\pi$.

(ii) Show similarly that $\mathrm{g}(x) > 0$ for $0 < x < \frac{1}{2}\pi$.

(iii) Show that $\mathrm{h}(x) > 0$ for $0 < x < \frac{1}{4}\pi$, and hence that

$$x(\sin^2 x + 3\cos^2 x) - 3\sin x\cos x > 0$$

for $0 < x < \frac{1}{4}\pi$.

(iv) By considering $\dfrac{\mathrm{F}'(x)}{\mathrm{F}(x)}$, show that $\mathrm{F}'(x) < 0$ for $0 < x < \frac{1}{4}\pi$.

STEP II 2007 Question 5 (Pure)

5 In this question, $\mathrm{f}^2(x)$ denotes $\mathrm{f}(\mathrm{f}(x))$, $\mathrm{f}^3(x)$ denotes $\mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$, and so on.

(i) The function f is defined, for $x \neq \pm 1/\sqrt{3}$, by

$$\mathrm{f}(x) = \frac{x+\sqrt{3}}{1-\sqrt{3}\,x}\,.$$

Find by direct calculation $\mathrm{f}^2(x)$ and $\mathrm{f}^3(x)$, and determine $\mathrm{f}^{2007}(x)$.

(ii) Show that $\mathrm{f}^n(x) = \tan(\theta + \frac{1}{3}n\pi)$, where $x = \tan\theta$ and n is any positive integer.

(iii) The function $\mathrm{g}(t)$ is defined, for $|t| \leqslant 1$ by $\mathrm{g}(t) = \frac{\sqrt{3}}{2}t + \frac{1}{2}\sqrt{1-t^2}$. Find an expression for $\mathrm{g}^n(t)$ for any positive integer n.

STEP II 2013 Question 5 (Pure)

5 **(i)** A function $\mathrm{f}(x)$ satisfies $\mathrm{f}(x) = \mathrm{f}(1-x)$ for all x. Show, by differentiating with respect to x, that $\mathrm{f}'(\frac{1}{2}) = 0$. If, in addition, $\mathrm{f}(x) = \mathrm{f}(\frac{1}{x})$ for all (non-zero) x, show that $\mathrm{f}'(-1) = 0$ and that $\mathrm{f}'(2) = 0$.

(ii) The function f is defined, for $x \neq 0$ and $x \neq 1$, by

$$\mathrm{f}(x) = \frac{(x^2 - x + 1)^3}{(x^2 - x)^2}.$$

Show that $\mathrm{f}(x) = \mathrm{f}(\frac{1}{x})$ and $\mathrm{f}(x) = \mathrm{f}(1-x)$.

Given that it has exactly three stationary points, sketch the curve $y = \mathrm{f}(x)$.

(iii) Hence, or otherwise, find all the roots of the equation $\mathrm{f}(x) = \dfrac{27}{4}$ and state the ranges of values of x for which $\mathrm{f}(x) > \dfrac{27}{4}$.

Find also all the roots of the equation $\mathrm{f}(x) = \dfrac{343}{36}$ and state the ranges of values of x for which $\mathrm{f}(x) > \dfrac{343}{36}$.

STEP I 2013 Question 8 (Pure)

8 **(i)** The functions a, b, c and d are defined by

$$\begin{aligned} \mathrm{a}(x) &= x^2 \quad (-\infty < x < \infty), \\ \mathrm{b}(x) &= \ln x \quad (x > 0), \\ \mathrm{c}(x) &= 2x \quad (-\infty < x < \infty), \\ \mathrm{d}(x) &= \sqrt{x} \quad (x \geqslant 0)\,. \end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

$$\mathrm{cb}, \qquad \mathrm{ab}, \qquad \mathrm{da}, \qquad \mathrm{ad}.$$

(ii) The functions f and g are defined by

$$\begin{aligned} \mathrm{f}(x) &= \sqrt{x^2 - 1} \quad (|x| \geqslant 1), \\ \mathrm{g}(x) &= \sqrt{x^2 + 1} \quad (-\infty < x < \infty). \end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned} \mathrm{h}(x) &= x + \sqrt{x^2 - 1} \quad (x \geqslant 1), \\ \mathrm{k}(x) &= x - \sqrt{x^2 - 1} \quad (|x| \geqslant 1), \end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

STEP II 2009 Question 2 (Pure)

2 The curve C has equation

$$y = a^{\sin(\pi e^x)},$$

where $a > 1$.

(i) Find the coordinates of the stationary points on C.

(ii) Use the approximations $e^t \approx 1 + t$ and $\sin t \approx t$ (both valid for small values of t) to show that

$$y \approx 1 - \pi x \ln a$$

for small values of x.

(iii) Sketch C.

(iv) By approximating C by means of straight lines joining consecutive stationary points, show that the area between C and the x-axis between the kth and $(k+1)$th maxima is approximately

$$\left(\frac{a^2+1}{2a}\right) \ln\left(1 + (k - \tfrac{3}{4})^{-1}\right).$$

STEP III 2009 Question 3 (Pure)

3 The function $\mathrm{f}(t)$ is defined, for $t \neq 0$, by

$$\mathrm{f}(t) = \frac{t}{\mathrm{e}^t - 1}\,.$$

(i) By expanding e^t, show that $\lim_{t\to 0} \mathrm{f}(t) = 1$. Find $\mathrm{f}'(t)$ and evaluate $\lim_{t\to 0} \mathrm{f}'(t)$.

(ii) Show that $\mathrm{f}(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $\mathrm{g}(t)$ is said to be *even* if $\mathrm{g}(t) \equiv \mathrm{g}(-t)$.]

(iii) Show with the aid of a sketch that $\mathrm{e}^t(1-t) \leqslant 1$ and deduce that $\mathrm{f}'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $\mathrm{f}(t)$.

STEP II 2013 Question 8 (Pure)

8 The function f satisfies $f(x) > 0$ for $x \geqslant 0$ and is strictly decreasing (which means that $f(b) < f(a)$ for $b > a$).

(i) For $t \geqslant 0$, let $A_0(t)$ be the area of the largest rectangle with sides parallel to the coordinate axes that can fit in the region bounded by the curve $y = f(x)$, the y-axis and the line $y = f(t)$. Show that $A_0(t)$ can be written in the form

$$A_0(t) = x_0\left(f(x_0) - f(t)\right),$$

where x_0 satisfies $x_0 f'(x_0) + f(x_0) = f(t)$.

(ii) The function g is defined, for $t > 0$, by

$$g(t) = \frac{1}{t}\int_0^t f(x)\,dx\,.$$

Show that $t g'(t) = f(t) - g(t)$.

Making use of a sketch show that, for $t > 0$,

$$\int_0^t \left(f(x) - f(t)\right) dx > A_0(t)$$

and deduce that $-t^2 g'(t) > A_0(t)$.

(iii) In the case $f(x) = \dfrac{1}{1+x}$, use the above to establish the inequality

$$\ln\sqrt{1+t} > 1 - \frac{1}{\sqrt{1+t}}\,,$$

for $t > 0$.

STEP II 2011 Question 3 (Pure)

3 In this question, you may assume without proof that any function f for which $\mathrm{f}'(x) \geqslant 0$ is *increasing*; that is, $\mathrm{f}(x_2) \geqslant \mathrm{f}(x_1)$ if $x_2 \geqslant x_1$.

(i) **(a)** Let $\mathrm{f}(x) = \sin x - x\cos x$. Show that $\mathrm{f}(x)$ is increasing for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and deduce that $\mathrm{f}(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(b) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \geqslant 1$ for $0 \leqslant x < 1$, show that

$$\arcsin x \geqslant x \qquad (0 \leqslant x < 1).$$

(c) Let $\mathrm{g}(x) = x \operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)\, x^{-1} \geqslant x \operatorname{cosec} x \qquad (0 < x < 1).$$

(ii) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arctan x) \leqslant 1$ for $x \geqslant 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geqslant x^2 \qquad (0 < x < \tfrac{1}{2}\pi).$$