STEP Past Papers by Topic

STEP Topic – Inequality

STEP I 2017 Question 2 (Pure)

2 **(i)** The inequality $\frac{1}{t} \leqslant 1$ holds for $t \geqslant 1$. By integrating both sides of this inequality over the interval $1 \leqslant t \leqslant x$, show that

$$\ln x \leqslant x - 1 \qquad (*)$$

for $x \geqslant 1$. Show similarly that $(*)$ also holds for $0 < x \leqslant 1$.

(ii) Starting from the inequality $\frac{1}{t^2} \leqslant \frac{1}{t}$ for $t \geqslant 1$, show that

$$\ln x \geqslant 1 - \frac{1}{x} \qquad (**)$$

for $x > 0$.

(iii) Show, by integrating $(*)$ and $(**)$, that

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y}$$

for $y > 0$ and $y \neq 1$.

STEP II 1987 Question 8 (Pure)

8 Show that, if the lengths of the diagonals of a parallelogram are specified, then the parallelogram has maximum area when the diagonals are perpendicular. Show also that the area of a parallelogram is less than or equal to half the square of the length of its longer diagonal.

The set A of points (x, y) is given by

$$|a_1x + b_1y - c_1| \leqslant \delta,$$
$$|a_2x + b_2y - c_2| \leqslant \delta,$$

with $a_1b_2 \neq a_2b_1$. Sketch this set and show that it is possible to find $(x_1, y_1), (x_2, y_2) \in A$ with

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 \geqslant \frac{8\delta^2}{|a_1b_2 - a_2b_1|}.$$

STEP II 1993 Question 8 (Pure)

8 Suppose that $a_i > 0$ for all $i > 0$. Show that

$$a_1 a_2 \leqslant \left(\frac{a_1 + a_2}{2} \right)^2 .$$

Prove by induction that for all positive integers m

$$a_1 \cdots a_{2^m} \leqslant \left(\frac{a_1 + \cdots + a_{2^m}}{2^m} \right)^{2^m} . \qquad (*)$$

If $n < 2^m$, put $b_1 = a_2$, $b_2 = a_2, \cdots, b_n = a_n$ and $b_{n+1} = \cdots = b_{2^m} = A$, where

$$A = \frac{a_1 + \cdots + a_n}{n}.$$

By applying $(*)$ to the b_i, show that

$$a_1 \cdots a_n A^{(2^m - n)} \leqslant A^{2^m}$$

(notice that $b_1 + \cdots + b_n = nA$). Deduce the (arithmetic mean)/(geometric mean) inequality

$$(a_1 \cdots a_n)^{1/n} \leqslant \frac{a_1 + \cdots + a_n}{n}.$$

STEP II 1991 Question 6 (Pure)

6 Show by means of a sketch, or otherwise, that if $0 \leqslant \mathrm{f}(y) \leqslant \mathrm{g}(y)$ for $0 \leqslant y \leqslant x$ then

$$0 \leqslant \int_0^x \mathrm{f}(y)\,\mathrm{d}y \leqslant \int_0^x \mathrm{g}(y)\,\mathrm{d}y.$$

Starting from the inequality $0 \leqslant \cos y \leqslant 1$, or otherwise, prove that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $0 \leqslant \sin x \leqslant x$ and $\cos x \geqslant 1 - \frac{1}{2}x^2$. Deduce that

$$\frac{1}{1800} \leqslant \int_0^{\frac{1}{10}} \frac{x}{(2+\cos x)^2}\,\mathrm{d}x \leqslant \frac{1}{1797}.$$

Show further that if $0 \leqslant x \leqslant \frac{1}{2}\pi$ then $\sin x \geqslant x - \frac{1}{6}x^3$. Hence prove that

$$\frac{1}{3000} \leqslant \int_0^{\frac{1}{10}} \frac{x^2}{(1-x+\sin x)^2}\,\mathrm{d}x \leqslant \frac{2}{5999}.$$

STEP II 1990 Question 6 (Pure)

6 Let a, b, c, d, p and q be positive integers. Prove that:

(i) if $b > a$ and $c > 1$, then $bc \geqslant 2c \geqslant 2 + c$;

(ii) if $a < b$ and $d < c$, then $bc - ad \geqslant a + c$;

(iii) if $\dfrac{a}{b} < p < \dfrac{c}{d}$, then $(bc - ad)\,p \geqslant a + c$;

(iv) if $\dfrac{a}{b} < \dfrac{p}{q} < \dfrac{c}{d}$, then $p \geqslant \dfrac{a+c}{bc-ad}$ and $q \geqslant \dfrac{b+d}{bc-ad}$.

Hence find all fractions with denominators less than 20 which lie between $8/9$ and $9/10$.

STEP III 1993 Question 9 (Pure)

9 For the real numbers $a_1, a_2, a_3, \ldots,$

(i) prove that $a_1^2 + a_2^2 \geqslant 2a_1a_2,$

(ii) prove that $a_1^2 + a_2^2 + a_3^2 \geqslant a_2a_3 + a_3a_1 + a_1a_2,$

(iii) prove that $3(a_1^2 + a_2^2 + a_3^2 + a_4^2) \geqslant 2(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4),$

(iv) state and prove a generalisation of (iii) to the case of n real numbers,

(v) prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geqslant \frac{2n}{n-1} \sum a_i a_j,$$

where the latter sum is taken over all pairs (i, j) with $1 \leqslant i < j \leqslant n$.

STEP III 2002 Question 4 (Pure)

4 Show that if x and y are positive and $x^3 + x^2 = y^3 - y^2$ then $x < y$.

Show further that if $0 < x \leqslant y - 1$, then $x^3 + x^2 < y^3 - y^2$.

Prove that there does not exist a pair of *positive* integers such that the difference of their cubes is equal to the sum of their squares.

Find all the pairs of integers such that the difference of their cubes is equal to the sum of their squares.

STEP I 1995 Question 1 (Pure)

1 **(i)** Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \geqslant 0.$$

(ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

(iii) On a sketch shade the regions of the (x, y) plane for which

$$x^3 - 4x^2y - xy^2 + 4y^3 \geqslant 0.$$

STEP II 2006 Question 3 (Pure)

3 **(i)** Show that $(5+\sqrt{24}\,)^4 + \dfrac{1}{(5+\sqrt{24}\,)^4}$ is an integer.

Show also that

$$0.1 < \frac{1}{5+\sqrt{24}} < \frac{2}{19} < 0.11\,.$$

Hence determine, with clear reasoning, the value of $(5+\sqrt{24})^4$ correct to four decimal places.

(ii) If N is an integer greater than 1, show that $(N+\sqrt{N^2-1}\,)^k$, where k is a positive integer, differs from the integer nearest to it by less than $\left(2N-\frac{1}{2}\right)^{-k}$.

STEP I 2012 Question 3 (Pure)

3 **(i)** Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and add to your diagram the tangent to the curve at the origin and the chord joining the origin to the point $(b, \sin b)$, where $0 < b < \frac{1}{2}\pi$.

By considering areas, show that

$$1 - \tfrac{1}{2}b^2 < \cos b < 1 - \tfrac{1}{2}b\sin b\,.$$

(ii) By considering the curve $y = a^x$, where $a > 1$, show that

$$\frac{2(a-1)}{a+1} < \ln a < -1 + \sqrt{2a-1}\,.$$

[**Hint**: You may wish to write a^x as $\mathrm{e}^{x\ln a}$.]

STEP I 2002 Question 3 (Pure)

3 Show that $(a+b)^2 \leqslant 2a^2 + 2b^2$.

Find the stationary points on the curve $y = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}}$, where a and b are constants. State, with brief reasons, which points are maxima and which are minima. Hence prove that

$$|a| + |b| \leqslant \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}} \leqslant \left(2a^2 + 2b^2\right)^{\frac{1}{2}}.$$

STEP III 2002 Question 7 (Pure)

7 Given that α and β are acute angles, show that $\alpha + \beta = \frac{1}{2}\pi$ if and only if $\cos^2 \alpha + \cos^2 \beta = 1$.

In the x–y plane, the point A has coordinates $(0, s)$ and the point C has coordinates $(s, 0)$, where $s > 0$. The point B lies in the first quadrant ($x > 0$, $y > 0$). The lengths of AB, OB and CB are respectively a, b and c.

Show that

$$(s^2 + b^2 - a^2)^2 + (s^2 + b^2 - c^2)^2 = 4s^2b^2$$

and hence that

$$(2s^2 - a^2 - c^2)^2 + (2b^2 - a^2 - c^2)^2 = 4a^2c^2 .$$

Deduce that

$$(a - c)^2 \leqslant 2b^2 \leqslant (a + c)^2 .$$

STEP I 1999 Question 6 (Pure)

6 **(i)** Find the greatest and least values of $bx + a$ for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.

(ii) Find the greatest and least values of $cx^2 + bx + a$, where $c \geqslant 0$, for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases that can arise for different values of b and c.
