# STEP Past Papers by Topic

# STEP Topic – Integration by substitution

**STEP I 1991 Question 8 (Pure)**

**8** **(i)** By a substitution of the form $y = k - x$ for suitable $k$, prove that, for any function $\mathrm{f}$,

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \pi \int_0^{\frac{1}{2}\pi} \mathrm{f}(\sin x)\,\mathrm{d}x.$$

Hence or otherwise evaluate

$$\int_0^{\pi} \frac{x}{2+\sin x}\,\mathrm{d}x.$$

**(ii)** Evaluate

$$\int_0^1 \frac{(\sin^{-1} t)\cos\left[(\sin^{-1} t)^2\right]}{\sqrt{1-t^2}}\,\mathrm{d}t.$$

[No credit will be given for numerical answers obtained by use of a calculator.]

**STEP I Specimen Question 4 (Pure)**

**4** Evaluate

$$\int_1^{\infty} \frac{1}{(x+1)\sqrt{x^2+2x-2}}\,\mathrm{d}x.$$

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**STEP I 1987 Question 5 (Pure)**

**5** Using the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$, show that, if $\alpha < \beta$,

$$\int_\alpha^\beta \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x = \pi.$$

What is the value of the above integral if $\alpha > \beta$?

Show also that, if $0 < \alpha < \beta$,

$$\int_\alpha^\beta \frac{1}{x\sqrt{(x-\alpha)(\beta-x)}}\,\mathrm{d}x = \frac{\pi}{\sqrt{\alpha\beta}}.$$

**STEP I 1993 Question 4 (Pure)**

**4** By making the change of variable $t = \pi - x$ in the integral

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x,$$

or otherwise, show that, for any function f,

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x\,.$$

Evaluate

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x \quad \text{and} \quad \int_0^{2\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x\,.$$

**STEP I 2017 Question 1 (Pure)**

**1** **(i)** Use the substitution $u = x\sin x + \cos x$ to find

$$\int \frac{x}{x\tan x + 1}\,\mathrm{d}x\,.$$

Find by means of a similar substitution, or otherwise,

$$\int \frac{x}{x\cot x - 1}\,\mathrm{d}x\,.$$

**(ii)** Use a substitution to find

$$\int \frac{x\sec^2 x\,\tan x}{x\sec^2 x - \tan x}\,\mathrm{d}x$$

and

$$\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2}\,\mathrm{d}x\,.$$

**STEP I 2004 Question 4 (Pure)**

**4** Differentiate $\sec t$ with respect to $t$.

**(i)** Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$ .

**(ii)** Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x$ .

**(iii)** Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x$ .

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**STEP I 1998 Question 2 (Pure)**

**2** Show, by means of a suitable change of variable, or otherwise, that

$$\int_0^{\infty} \mathrm{f}((x^2+1)^{1/2}+x)\,\mathrm{d}x = \frac{1}{2}\int_1^{\infty}(1+t^{-2})\mathrm{f}(t)\,\mathrm{d}t.$$

Hence, or otherwise, show that

$$\int_0^{\infty}((x^2+1)^{1/2}+x)^{-3}\,\mathrm{d}x = \frac{3}{8}.$$

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**STEP III 2000 Question 2 (Pure)**

**2** Use the substitution $x = 2 - \cos\theta$ to evaluate the integral

$$\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{\frac{1}{2}} \mathrm{d}x.$$

Show that, for $a < b$,

$$\int_{p}^{q} \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} \mathrm{d}x = \frac{(b-a)(\pi + 3\sqrt{3} - 6)}{12},$$

where $p = (3a+b)/4$ and $q = (a+b)/2$.

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**STEP I 2006 Question 5 (Pure)**

**5** **(i)** Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}}\,\mathrm{d}x = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K\,,$$

where $K$ is a constant.

**(ii)** Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\,\mathrm{d}x = \frac{7}{12} + \ln\frac{2}{3}\,.$

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## STEP III 2005 Question 7 (Pure)

**7** Show that if $\int \frac{1}{u\,\mathrm{f}(u)}\,\mathrm{d}u = \mathrm{F}(u) + c$, then $\int \frac{m}{x\,\mathrm{f}(x^m)}\,\mathrm{d}x = \mathrm{F}(x^m) + c$, where $m \neq 0$.

Find:

**(i)** $\int \frac{1}{x^n - x}\,\mathrm{d}x$;

**(ii)** $\int \frac{1}{\sqrt{x^n + x^2}}\,\mathrm{d}x$.

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## STEP III 2004 Question 1 (Pure)

**1** Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x\,.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right).$$

By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u\,.$$

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**STEP II 2007 Question 3 (Pure)**

**3** By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}$.

**(i)** Let $I = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x$.

**(a)** Evaluate $I$.

**(b)** Use the substitution $t = \tan\frac{1}{2}x$ to show that $\displaystyle\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t = \tfrac{1}{2}I$.

**(ii)** Evaluate $\displaystyle\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,\mathrm{d}t$.

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**STEP I 1994 Question 8 (Pure)**

**8** By means of the change of variable $\theta = \frac{1}{4}\pi - \phi$, or otherwise, show that

$$\int_0^{\frac{1}{4}\pi} \ln(1 + \tan\theta)\,\mathrm{d}\theta = \tfrac{1}{8}\pi \ln 2.$$

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{1}{2}\pi} \ln\left(\frac{1+\sin x}{1+\cos x}\right)\,\mathrm{d}x.$$

**STEP II 2012 Question 3 (Pure)**

**3** Show that, for any function f (for which the integrals exist),

$$\int_0^\infty \mathrm{f}\big(x + \sqrt{1+x^2}\big)\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \left(1 + \frac{1}{t^2}\right)\mathrm{f}(t)\,\mathrm{d}t\,.$$

Hence evaluate

$$\int_0^\infty \frac{1}{2x^2+1+2x\sqrt{x^2+1}}\,\mathrm{d}x\,,$$

and, using the substitution $x = \tan\theta$,

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta\,.$$

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**STEP II 2006 Question 4 (Pure)**

**4** By making the substitution $x = \pi - t$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \tfrac{1}{2}\pi \int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x\,,$$

where $\mathrm{f}(\sin x)$ is a given function of $\sin x$.

Evaluate the following integrals:

**(i)** $\displaystyle\int_0^{\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

**(ii)** $\displaystyle\int_0^{2\pi} \frac{x\sin x}{3+\sin^2 x}\,\mathrm{d}x\,;$

**(iii)** $\displaystyle\int_0^{\pi} \frac{x|\sin 2x|}{3+\sin^2 x}\,\mathrm{d}x\,.$

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**STEP II 2014 Question 4 (Pure)**

**4** **(i)** By using the substitution $u = 1/x$, show that for $b > 0$

$$\int_{1/b}^{b} \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)} \, dx = 0 \, .$$

**(ii)** By using the substitution $u = 1/x$, show that for $b > 0$,

$$\int_{1/b}^{b} \frac{\arctan x}{x} \, dx = \frac{\pi \ln b}{2} \, .$$

**(iii)** By using the result $\int_0^\infty \frac{1}{a^2 + x^2} \, dx = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = k/x$, for suitable $k$, show that

$$\int_0^\infty \frac{1}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a^3} \qquad (a > 0).$$

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**STEP I 2010 Question 4 (Pure)**

**4** Use the substitution $x = \dfrac{1}{t^2 - 1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\,\right) + c\,.$$

**[Note** You may use without proof the result $\displaystyle\int \frac{1}{t^2 - a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$ **]**

The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through $360^o$ about the $x$-axis. Show that the volume enclosed is $2\pi\ln\frac{5}{4}$.

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**STEP I 2009 Question 6 (Pure)**

**6** **(i)** Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

**(ii)** Show by means of a substitution that

$$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

**(iii)** Evaluate:

**(a)** $\displaystyle\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x$ ;

**(b)** $\displaystyle\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x$ .

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**STEP II 2010 Question 4 (Pure)**

**4** **(i)** Let

$$I = \int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x\,.$$

Use a substitution to show that

$$I = \int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x$$

and hence evaluate $I$ in terms of $a$.

Use this result to evaluate the integrals

$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin(x+\frac{\pi}{4})}\,\mathrm{d}x\,.$$

**(ii)** Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{\sin x}{x\big(\sin x + \sin\frac{1}{x}\big)}\,\mathrm{d}x\,.$$

**STEP III 2014 Question 2 (Pure)**

**2** **(i)** Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x}\,\mathrm{d}x = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C\,.$$

**(ii)** Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x}\,\mathrm{d}x\,.$$

**(iii)** Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1+u^4}\,\mathrm{d}u = \frac{\pi + 2\ln(\sqrt{2}+1)}{4\sqrt{2}}\,.$$

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**STEP III 2007 Question 7 (Pure)**

**7** The functions $\mathrm{s}(x)$ $(0 \leqslant x < 1)$ and $\mathrm{t}(x)$ $(x \geqslant 0)$, and the real number $p$, are defined by

$$\mathrm{s}(x) = \int_0^x \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u\,, \quad \mathrm{t}(x) = \int_0^x \frac{1}{1+u^2}\,\mathrm{d}u\,, \quad p = 2\int_0^\infty \frac{1}{1+u^2}\,\mathrm{d}u\,.$$

For this question, do not evaluate any of the above integrals explicitly in terms of inverse trigonometric functions or the number $\pi$.

**(i)** Use the substitution $u = v^{-1}$ to show that $\mathrm{t}(x) = \displaystyle\int_{1/x}^\infty \frac{1}{1+v^2}\,\mathrm{d}v$. Hence evaluate $\mathrm{t}(1/x) + \mathrm{t}(x)$ in terms of $p$ and deduce that $2\mathrm{t}(1) = \frac{1}{2}p$.

**(ii)** Let $y = \dfrac{u}{\sqrt{1+u^2}}$. Express $u$ in terms of $y$, and show that $\dfrac{\mathrm{d}u}{\mathrm{d}y} = \dfrac{1}{\sqrt{(1-y^2)^3}}$.

By making a substitution in the integral for $\mathrm{t}(x)$, show that

$$\mathrm{t}(x) = \mathrm{s}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Deduce that $\mathrm{s}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}p$.

**(iii)** Let $z = \dfrac{u + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}u}$. Show that $\mathrm{t}\left(\frac{1}{\sqrt{3}}\right) = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+z^2}\,\mathrm{d}z$, and hence that $3\mathrm{t}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}p$.

**STEP I 2013 Question 4 (Pure)**

**4** **(i)** Show that, for $n > 0$,

$$\int_0^{\frac{1}{4}\pi} \tan^n x \sec^2 x \,\mathrm{d}x = \frac{1}{n+1} \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} \sec^n x \tan x \,\mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n}\,.$$

**(ii)** Evaluate the following integrals:

$$\int_0^{\frac{1}{4}\pi} x \sec^4 x \tan x \,\mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \tan x \,\mathrm{d}x\,.$$

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