STEPPast Papers by Topic

STEP Topic - Integration

STEP I 1987 Question 6 (Pure)

6 Let $y = \mathrm{f}(x)$, $(0 \leqslant x \leqslant a)$, be a continuous curve lying in the first quadrant and passing through the origin. Suppose that, for each non-negative value of y with $0 \leqslant y \leqslant \mathrm{f}(a)$, there is *exactly* one value of x such that $\mathrm{f}(x) = y$; thus we may write $x = \mathrm{g}(y)$, for a suitable function g.

For $0 \leqslant s \leqslant a$, $0 \leqslant t \leqslant \mathrm{f}(a)$, define

$$\mathrm{F}(s) = \int_0^s \mathrm{f}(x)\,\mathrm{d}x, \qquad \mathrm{G}(t) = \int_0^t \mathrm{g}(y)\,\mathrm{d}y.$$

By a geometrical argument, show that

$$\mathrm{F}(s) + \mathrm{G}(t) \geqslant st. \qquad (*)$$

When does equality occur in $(*)$?

Suppose that $y = \sin x$ and that the ranges of x, y, s, t are restricted to $0 \leqslant x \leqslant s \leqslant \frac{1}{2}\pi$, $0 \leqslant y \leqslant t \leqslant 1$. By considering s such that the equality holds in $(*)$, show that

$$\int_0^t \sin^{-1} y\,\mathrm{d}y = t\sin^{-1} t - \left(1 - \cos(\sin^{-1} t)\right).$$

Check this result by differentiating both sides with respect to t.

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STEP III 1992 Question 6 (Pure)

6 Given that $I_n = \displaystyle\int_0^\pi \frac{x\sin^2(nx)}{\sin^2 x}\,\mathrm{d}x$, where n is a positive integer, show that $I_n - I_{n-1} = J_n$, where

$$J_n = \int_0^\pi \frac{x\sin(2n-1)x}{\sin x}\,\mathrm{d}x.$$

Obtain also a reduction formula for J_n.

The curve C is given by the cartesian equation

$$y = \frac{x\sin^2(nx)}{\sin^2 x},$$

where n is a positive integer and $0 \leqslant x \leqslant \pi$. Show that the area under the curve C is $\frac{1}{2}n\pi^2$.

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STEP II 1989 Question 6 (Pure)

6 The function f satisfies the condition $\mathrm{f}'(x) > 0$ for $a \leqslant x \leqslant b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta \mathrm{g}(y)\,\mathrm{d}y,$$

where $\alpha = \mathrm{f}(a)$ and $\beta = \mathrm{f}(b)$. Interpret this formula geometrically, in the case where α and a are both positive.

Prove similarly and interpret (for $\alpha > 0$ and $a > 0$) the formula

$$2\pi \int_a^b x\mathrm{f}(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [\mathrm{g}(y)]^2\ \mathrm{d}y.$$

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STEP II 1987 Question 7 (Pure)

7 A definite integral can be evaluated approximately by means of the Trapezium rule:

$$\int_{x_0}^{x_N} \mathrm{f}(x)\,\mathrm{d}x \approx \tfrac{1}{2}h\left\{\mathrm{f}(x_0) + 2\mathrm{f}(x_1) + \ldots + 2\mathrm{f}(x_{N-1}) + \mathrm{f}(x_N)\right\},$$

where the interval length h is given by $Nh = x_N - x_0$, and $x_r = x_0 + rh$. Justify briefly this approximation.

Use the Trapezium rule with intervals of unit length to evaluate approximately the integral

$$\int_1^n \ln x\,\mathrm{d}x,$$

where $n(>2)$ is an integer. Deduce that $n! \approx \mathrm{g}(n)$, where

$$\mathrm{g}(n) = n^{n+\frac{1}{2}}\mathrm{e}^{1-n},$$

and show by means of a sketch, or otherwise, that

$$n! < \mathrm{g}(n).$$

By using the Trapezium rule on the above integral with intervals of width k^{-1}, where k is a positive integer, show that

$$(kn)! \approx k!\,n^{kn+\frac{1}{2}}\left(\frac{\mathrm{e}}{k}\right)^{k(1-n)}.$$

Determine whether this approximation or $\mathrm{g}(kn)$ is closer to $(kn)!$.

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STEP II 1991 Question 5 (Pure)

5 Give a rough sketch of the function $\tan^k \theta$ for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$ in the two cases $k = 1$ and $k \gg 1$ (i.e. k is much greater than 1).

Show that for any positive integer n

$$\int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \, \mathrm{d}\theta = (-1)^n \left(\tfrac{1}{2}\ln 2 + \sum_{m=1}^{n} \frac{(-1)^m}{2m} \right),$$

and deduce that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m} = \tfrac{1}{2}\ln 2.$$

Show similarly that

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} = \frac{\pi}{4}.$$

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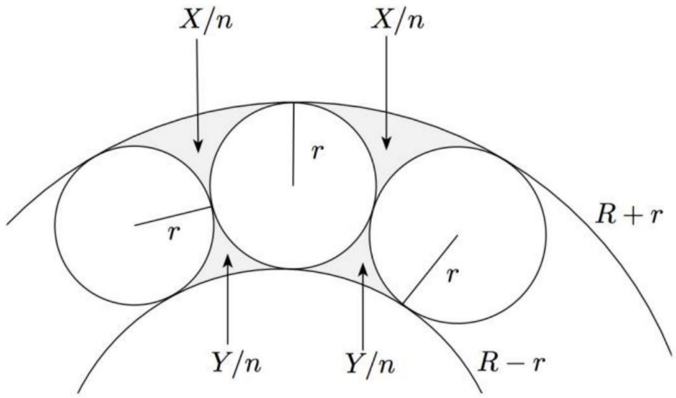
STEP I 1987 Question 2 (Pure)

2



The region A between concentric circles of radii $R+r$, $R-r$ contains n circles of radius r. Each circle of radius r touches both of the larger circles as well as its two neighbours of radius r, as shown in the figure. Find the relationship which must hold between n, R and r.

Show that Y, the total area of A outside the circle of radius r and adjacent to the circle of radius $R-r$, is given by

$$Y = nr\sqrt{R^2 - r^2} - \pi(R-r)^2 - n\pi r^2\left(\frac{1}{2} - \frac{1}{n}\right).$$

Find similar expressions for X, the total area of A outside the circles of radius r and adjacent to the circle of radius $R+r$, and for Z, the total area inside the circle of radius r.

What value does $(X+Y)/Z$ approach when n becomes large?

STEP I 1989 Question 2 (Pure)

2 For $x > 0$ find $\int x \ln x \, \mathrm{d}x$.

By approximating the area corresponding to $\int_0^1 x \ln(1/x) \, \mathrm{d}x$ by n rectangles of equal width and with their top right-hand vertices on the curve $y = x \ln(1/x)$, show that, as $n \to \infty$,

$$\frac{1}{2}\left(1 + \frac{1}{n}\right) \ln n - \frac{1}{n^2}\left[\ln\left(\frac{n!}{0!}\right) + \ln\left(\frac{n!}{1!}\right) + \ln\left(\frac{n!}{2!}\right) + \cdots + \ln\left(\frac{n!}{(n-1)!}\right)\right] \to \frac{1}{4}.$$

[You may assume that $x \ln x \to 0$ as $x \to 0$.]

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STEP III 1991 Question 7 (Pure)

7 **(i)** Prove that

$$\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\cos x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x - \tfrac{1}{4}\pi\ln 2$$

and

$$\int_0^{\frac{1}{2}\pi} \ln(\sin 2x)\,\mathrm{d}x = \tfrac{1}{2}\int_0^{\pi} \ln(\sin x)\,\mathrm{d}x = \int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x.$$

Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} \ln(\sin x)\,\mathrm{d}x$.

[You may assume that all the integrals converge.]

(ii) Given that $\ln u < u$ for $u \geqslant 1$ deduce that

$$\tfrac{1}{2}\ln x < \sqrt{x} \qquad \text{for} \qquad x \geqslant 1.$$

Deduce that $\dfrac{\ln x}{x} \to 0$ as $x \to \infty$ and that $x\ln x \to 0$ as $x \to 0$ through positive values.

(iii) Using the results of parts **(i)** and **(ii)**, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}\pi} x\cot x\,\mathrm{d}x$.

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STEP II 2004 Question 5 (Pure)

5 Evaluate $\int_0^{\pi} x\sin x\,\mathrm{d}x$ and $\int_0^{\pi} x\cos x\,\mathrm{d}x$.

The function f satisfies the equation

$$\mathrm{f}(t) = t + \int_0^{\pi} \mathrm{f}(x)\sin(x+t)\,\mathrm{d}x\,. \qquad (*)$$

Show that

$$\mathrm{f}(t) = t + A\sin t + B\cos t\,,$$

where $A = \int_0^{\pi} \mathrm{f}(x)\cos x\,\mathrm{d}x$ and $B = \int_0^{\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x$.

Find A and B by substituting for $\mathrm{f}(t)$ and $\mathrm{f}(x)$ in $(*)$ and equating coefficients of $\sin t$ and $\cos t$.

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STEP I 2006 Question 7 (Pure)

7 **(i)** Sketch on the same axes the functions $\operatorname{cosec} x$ and $2x/\pi$, for $0 < x < \pi$. Deduce that the equation $x \sin x = \pi/2$ has exactly two roots in the interval $0 < x < \pi$.

Show that

$$\int_{\pi/2}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \, \mathrm{d}x = 2 \sin \alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha \cos \alpha - 1$$

where α is the larger of the roots referred to above.

(ii) Show that the region bounded by the positive x-axis, the y-axis and the curve

$$y = \Big| |\mathrm{e}^x - 1| - 1 \Big|$$

has area $\ln 4 - 1$.

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STEP I 2000 Question 3 (Pure)

3 For any number x, the largest integer less than or equal to x is denoted by $[x]$. For example, $[3.7] = 3$ and $[4] = 4$.

Sketch the graph of $y = [x]$ for $0 \leqslant x < 5$ and evaluate

$$\int_0^5 [x] \, \mathrm{d}x.$$

Sketch the graph of $y = [\mathrm{e}^x]$ for $0 \leqslant x < \ln n$, where n is an integer, and show that

$$\int_0^{\ln n} [\mathrm{e}^x] \, \mathrm{d}x = n \ln n - \ln(n!).$$

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STEP II 2000 Question 5 (Pure)

5 It is required to approximate a given function $\mathrm{f}(x)$, over the interval $0 \leqslant x \leqslant 1$, by the linear function λx, where λ is chosen to minimise

$$\int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \,\mathrm{d}x.$$

Show that

$$\lambda = 3\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x.$$

The residual error, R, of this approximation process is such that

$$R^2 = \int_0^1 \big(\mathrm{f}(x) - \lambda x\big)^2 \,\mathrm{d}x.$$

Show that

$$R^2 = \int_0^1 \big(\mathrm{f}(x)\big)^2 \,\mathrm{d}x - \tfrac{1}{3}\lambda^2.$$

Given now that $\mathrm{f}(x) = \sin(\pi x/n)$, show that **(i)** for large n, $\lambda \approx \pi/n$ and **(ii)** $\lim_{n\to\infty} R = 0$.

Explain why, prior to any calculation, these results are to be expected.

[You may assume that, when θ is small, $\sin\theta \approx \theta - \frac{1}{6}\theta^3$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.]

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STEP II 2003 Question 6 (Pure)

6 The function f is defined by

$$\mathrm{f}(x) = |x - 1| \,,$$

where the domain is $\mathbf{R}$, the set of all real numbers. The function $\mathrm{g}_n = \mathrm{f}^n$, with domain $\mathbf{R}$, so for example $\mathrm{g}_3(x) = \mathrm{f}(\mathrm{f}(\mathrm{f}(x)))$. In separate diagrams, sketch graphs of g_1, g_2, g_3 and g_4.

The function h is defined by

$$\mathrm{h}(x) = \left|\sin \frac{\pi x}{2}\right| \,,$$

where the domain is $\mathbf{R}$. Show that if n is even,

$$\int_0^n \big(\mathrm{h}(x) - \mathrm{g}_n(x)\big)\,\mathrm{d}x = \frac{2n}{\pi} - \frac{n}{2} \,.$$

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STEP III 2003 Question 1 (Pure)

1 Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin\left(\frac{x+a}{x+b}\right) = \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcosh}\left(\dfrac{x+b}{x+a}\right)$.

Hence, or otherwise, integrate, for $x > -1$,

(i) $\displaystyle\int \frac{1}{(x+1)\sqrt{x+3}}\,\mathrm{d}x$,

(ii) $\displaystyle\int \frac{1}{(x+3)\sqrt{x+1}}\,\mathrm{d}x$.

[You may use the results $\dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x} \operatorname{arcosh} x = \dfrac{1}{\sqrt{x^2-1}}$.]

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STEP I 2012 Question 5 (Pure)

5 Show that

$$\int_0^{\frac{1}{4}\pi} \sin(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{4}(\ln 2 - 1)\,,$$

and that

$$\int_0^{\frac{1}{4}\pi} \cos(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{8}(\pi - \ln 4 - 2)\,.$$

Hence evaluate

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \big(\cos(2x) + \sin(2x)\big)\;\ln\big(\cos x + \sin x\big)\,\mathrm{d}x\,.$$

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STEP I 2000 Question 4 (Pure)

4 **(i)** Show that, for $0 \leqslant x \leqslant 1$, the largest value of $\dfrac{x^6}{(x^2+1)^4}$ is $\dfrac{1}{16}$.

(ii) Find constants A, B, C and D such that, for all x,

$$\frac{1}{(x^2+1)^4} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{Dx^6}{(x^2+1)^4}.$$

(iii) Hence, or otherwise, prove that

$$\frac{11}{24} \leqslant \int_0^1 \frac{1}{(x^2+1)^4}\,\mathrm{d}x \leqslant \frac{11}{24} + \frac{1}{16}\,.$$

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STEP II 2010 Question 8 (Pure)

8 The curves C_1 and C_2 are defined by

$$y = e^{-x} \quad (x > 0) \quad \text{and} \quad y = e^{-x}\sin x \quad (x > 0),$$

respectively. Sketch roughly C_1 and C_2 on the same diagram.

Let x_n denote the x-coordinate of the nth point of contact between the two curves, where $0 < x_1 < x_2 < \cdots$, and let A_n denote the area of the region enclosed by the two curves between x_n and x_{n+1}. Show that

$$A_n = \tfrac{1}{2}(e^{2\pi} - 1)e^{-(4n+1)\pi/2}$$

and hence find $\displaystyle\sum_{n=1}^{\infty} A_n$.

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STEP II 2005 Question 3 (Pure)

3 Give a sketch, for $0 \leqslant x \leqslant \frac{1}{2}\pi$, of the curve

$$y = (\sin x - x\cos x) \ ,$$

and show that $0 \leqslant y \leqslant 1$.

Show that:

(i) $$\int_0^{\frac{1}{2}\pi} y \, \mathrm{d}x = 2 - \frac{\pi}{2} \ ;$$

(ii) $$\int_0^{\frac{1}{2}\pi} y^2 \, \mathrm{d}x = \frac{\pi^3}{48} - \frac{\pi}{8} \ .$$

Deduce that $\pi^3 + 18\pi < 96$.

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STEP II 2015 Question 6 (Pure)

6 **(i)** Show that

$$\sec^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) = \frac{2}{1+\sin x}\,.$$

Hence integrate $\dfrac{1}{1+\sin x}$ with respect to x.

(ii) By means of the substitution $y = \pi - x$, show that

$$\int_0^{\pi} x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \mathrm{f}(\sin x)\,\mathrm{d}x,$$

where f is any function for which these integrals exist.

Hence evaluate

$$\int_0^{\pi} \frac{x}{1+\sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^{\pi} \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x.$$

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STEP III 2015 Question 1 (Pure)

1 **(i)** Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u\,,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}\,.$$

(ii) Let

$$J = \int_0^\infty \mathrm{f}\big((x - x^{-1})^2\big)\,\mathrm{d}x\,,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2}\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \frac{1}{2}\int_0^\infty (1+x^{-2})\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x = \int_0^\infty \mathrm{f}\big(u^2\big)\,\mathrm{d}u\,.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4 - x^2 + 1)^n}\,\mathrm{d}x\,,$$

where n is a positive integer.

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STEP II 1994 Question 4 (Pure)

4 By considering the area of the region defined in terms of Cartesian coordinates (x, y) by

$$\{(x,y):\ x^2+y^2=1,\ 0\leqslant y,\ 0\leqslant x\leqslant c\},$$

show that

$$\int_0^c (1-x^2)^{\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{2}[c(1-c^2)^{\frac{1}{2}}+\sin^{-1}c],$$

if $0<c\leqslant 1$.

Show that the area of the region defined by

$$\left\{(x,y):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0\leqslant y,\ 0\leqslant x\leqslant c\right\},$$

is

$$\frac{ab}{2}\left[\frac{c}{a}\left(1-\frac{c^2}{a^2}\right)^{\frac{1}{2}}+\sin^{-1}\left(\frac{c}{a}\right)\right],$$

if $0<c\leqslant a$ and $0<b$.

Suppose that $0<b\leqslant a$. Show that the area of intersection $E\cap F$ of the two regions defined by

$$E=\left\{(x,y):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1\right\} \quad \text{and} \quad F=\left\{(x,y):\ \frac{x^2}{b^2}+\frac{y^2}{a^2}\leqslant 1\right\}$$

is

$$4ab\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right).$$

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STEP III 2011 Question 5 (Pure)

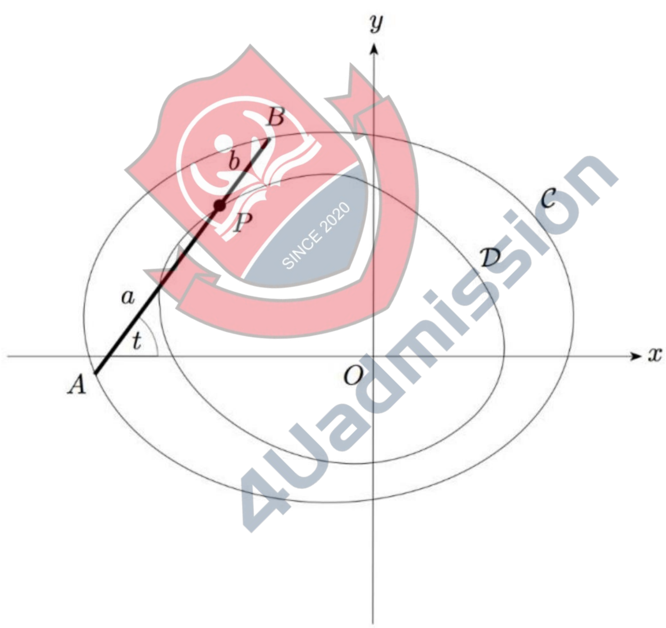
5 A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2\,\mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t\,. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



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Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi}\left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t}\right)\cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\sin t\right)\mathrm{d}t\,.$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

STEP I 2017 Question 6 (Pure)

6 In this question, you may assume that, if a continuous function takes both positive and negative values in an interval, then it takes the value 0 at some point in that interval.

(i) The function f is continuous and $\mathrm{f}(x)$ is non-zero for some value of x in the interval $0 \leqslant x \leqslant 1$. Prove by contradiction, or otherwise, that if

$$\int_0^1 \mathrm{f}(x)\mathrm{d}x = 0\,,$$

then $\mathrm{f}(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$.

(ii) The function g is continuous and

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x = 1\,, \quad \int_0^1 x\mathrm{g}(x)\,\mathrm{d}x = \alpha\,, \quad \int_0^1 x^2\mathrm{g}(x)\,\mathrm{d}x = \alpha^2\,. \qquad (*)$$

Show, by considering

$$\int_0^1 (x-\alpha)^2\mathrm{g}(x)\,\mathrm{d}x\,,$$

that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

Find a function of the form $\mathrm{g}(x) = a + bx$ that satisfies the conditions $(*)$ and verify that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

(iii) The function h has a continuous derivative h′ and

$$\mathrm{h}(0) = 0\,, \quad \mathrm{h}(1) = 1\,, \quad \int_0^1 \mathrm{h}(x)\,\mathrm{d}x = \beta\,, \quad \int_0^1 x\mathrm{h}(x)\,\mathrm{d}x = \tfrac{1}{2}\beta(2-\beta)\,.$$

Use the result in part **(ii)** to show that $\mathrm{h}'(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

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STEP I 1996 Question 2 (Pure)

2 **(i)** Show that

$$\int_0^1 (1+(\alpha-1)x)^n \, \mathrm{d}x = \frac{\alpha^{n+1}-1}{(n+1)(\alpha-1)}$$

when $\alpha \neq 1$ and n is a positive integer.

(ii) Show that if $0 \leqslant k \leqslant n$ then the coefficient of α^k in the polynomial

$$\int_0^1 (\alpha x + (1-x))^n \, \mathrm{d}x$$

is

$$\binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \, \mathrm{d}x \, .$$

(iii) Hence, or otherwise, show that

$$\int_0^1 x^k (1-x)^{n-k} \, \mathrm{d}x = \frac{k!(n-k)!}{(n+1)!} \, .$$

STEP III 2013 Question 1 (Pure)

1 Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0<a<1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

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STEP III 2016 Question 3 (Pure Mathematics)

3 **(i)** Given that

$$\int \frac{x^3 - 2}{(x+1)^2}\, \mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\, \mathrm{e}^x + \text{constant}\,,$$

where $\mathrm{P}(x)$and $\mathrm{Q}(x)$ are polynomials, show that $\mathrm{Q}(x)$ has a factor of $x+1$.

Show also that the degree of $\mathrm{P}(x)$ is exactly one more than the degree of $\mathrm{Q}(x)$, and find $\mathrm{P}(x)$ in the case $\mathrm{Q}(x) = x+1$.

(ii) Show that there are no polynomials $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ such that

$$\int \frac{1}{x+1}\, \mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\, \mathrm{e}^x + \text{constant}\,.$$

You need consider only the case when $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ have no common factors.

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STEP I 1994 Question 4 (Pure)

4 Show that

(i) $\dfrac{1-\cos\alpha}{\sin\alpha} = \tan\tfrac{1}{2}\alpha,$

(ii) if $|k| < 1$ then $\displaystyle\int \frac{\mathrm{d}x}{1-2kx+x^2} = \frac{1}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{x-k}{\sqrt{1-k^2}}\right) + C$, where C is a constant of integration.

Hence, or otherwise, show that if $0 < \alpha < \pi$ then

$$\int_0^1 \frac{\sin\alpha}{1-2x\cos\alpha+x^2}\,\mathrm{d}x = \frac{\pi-\alpha}{2}.$$

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STEP II 2009 Question 7 (Pure)

7 Let $y = (x-a)^n e^{bx}\sqrt{1+x^2}$, where n and a are constants and b is a non-zero constant. Show that

$$\frac{dy}{dx} = \frac{(x-a)^{n-1}e^{bx}q(x)}{\sqrt{1+x^2}},$$

where $q(x)$ is a cubic polynomial.

Using this result, determine:

(i) $$\int \frac{(x-4)^{14}e^{4x}(4x^3-1)}{\sqrt{1+x^2}}\,dx\,;$$

(ii) $$\int \frac{(x-1)^{21}e^{12x}(12x^4-x^2-11)}{\sqrt{1+x^2}}\,dx\,;$$

(iii) $$\int \frac{(x-2)^{6}e^{4x}(4x^4+x^3-2)}{\sqrt{1+x^2}}\,dx\,.$$

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STEP III 2010 Question 2 (Pure)

2 In this question, a is a positive constant.

(i) Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,\mathrm{d}x = \frac{a}{2\sinh a}\,.$$

(ii) Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,\mathrm{d}x$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,\mathrm{d}x\,.$$

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STEP III 2006 Question 2 (Pure)

2 Let

$$I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1 - \sin\theta\sin 2\alpha}\,\mathrm{d}\theta \quad \text{and} \quad J = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sec^2\theta}{1 + \tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$$

where $0 < \alpha < \frac{1}{4}\pi$.

(i) Show that $I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta}{1 + \sin\theta\sin 2\alpha}\,\mathrm{d}\theta$ and hence that $2I = \displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2}{1 + \tan^2\theta\cos^2 2\alpha}\,\mathrm{d}\theta$.

(ii) Find J.

(iii) By considering $I\sin^2 2\alpha + J\cos^2 2\alpha$, or otherwise, show that $I = \frac{1}{2}\pi\sec^2\alpha$.

(iv) Evaluate I in the case $\frac{1}{4}\pi < \alpha < \frac{1}{2}\pi$.

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STEP I 2014 Question 2 (Pure)

2 **(i)** Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c$, where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}$.

(iv) The curve B is given by $y = \big|\ln|x^2-4|\big|$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

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STEP I 1995 Question 2 (Pure)

2 **(i)** Suppose that

$$S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x \quad \text{and} \quad T = \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

By considering $S + T$ and $S - T$ determine S and T.

(ii) Evaluate $\int_{\frac{1}{4}}^{\frac{1}{2}} (1-4x)\sqrt{\frac{1}{x} - 1}\,\mathrm{d}x$ by using the substitution $x = \sin^2 t$.

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STEP I 2007 Question 3 (Pure)

3 Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence or otherwise evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^4\theta \,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^4\theta \,\mathrm{d}\theta\,.$$

Evaluate also

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^6\theta \,\mathrm{d}\theta\,.$$

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STEP I 2005 Question 5 (Pure)

5 **(i)** Evaluate the integral

$$\int_0^1 (x+1)^{k-1}\ \mathrm{d}x$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

(ii) Evaluate the integral

$$\int_0^1 x\,(x+1)^m\ \mathrm{d}x$$

in the different cases that arise according to the value of m.

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STEP II 2013 Question 2 (Pure)

2 For $n \geqslant 0$, let

$$I_n = \int_0^1 x^n(1-x)^n \mathrm{d}x\,.$$

(i) For $n \geqslant 1$, show by means of a substitution that

$$\int_0^1 x^{n-1}(1-x)^n \mathrm{d}x = \int_0^1 x^n(1-x)^{n-1}\mathrm{d}x$$

and deduce that

$$2\int_0^1 x^{n-1}(1-x)^n \mathrm{d}x = I_{n-1}\,.$$

Show also, for $n \geqslant 1$, that

$$I_n = \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1}\mathrm{d}x$$

and hence that $I_n = \dfrac{n}{2(2n+1)}I_{n-1}$.

(ii) When n is a positive integer, show that

$$I_n = \frac{(n!)^2}{(2n+1)!}\,.$$

(iii) Use the substitution $x = \sin^2\theta$ to show that $I_{\frac{1}{2}} = \frac{\pi}{8}$, and evaluate $I_{\frac{3}{2}}$.

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STEP II 2010 Question 2 (Pure)

2 Prove that

$$\cos 3x = 4\cos^3 x - 3\cos x\,.$$

Find and prove a similar result for $\sin 3x$ in terms of $\sin x$.

(i) Let

$$\mathrm{I}(\alpha) = \int_0^\alpha \big(7\sin x - 8\sin^3 x\big)\mathrm{d}x\,.$$

Show that

$$\mathrm{I}(\alpha) = -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}\,,$$

where $c = \cos\alpha$. Write down one value of c for which $\mathrm{I}(\alpha) = 0$.

(ii) Useless Eustace believes that

$$\int \sin^n x\,\mathrm{d}x = \frac{\sin^{n+1} x}{n+1}$$

for $n = 1,\ 2,\ 3, \ldots$. Show that Eustace would obtain the correct value of $\mathrm{I}(\beta)$, where $\cos\beta = -\frac{1}{6}$.

Find all values of α for which he would obtain the correct value of $\mathrm{I}(\alpha)$.

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STEP II 2009 Question 5 (Pure)

5 Expand and simplify $(\sqrt{x-1}+1)^2$.

(i) Evaluate

$$\int_5^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}\,\mathrm{d}x\,.$$

(ii) Find the total area between the curve

$$y=\frac{\sqrt{x-2\sqrt{x-1}}}{\sqrt{x-1}}$$

and the x-axis between the points $x=\frac{5}{4}$ and $x=10$.

(iii) Evaluate

$$\int_{\frac{5}{4}}^{10} \frac{\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x+1}+2}}{\sqrt{x^2-1}}\,\mathrm{d}x\,.$$

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STEP II 2011 Question 6 (Pure)

6 For any given function f, let

$$I = \int [\mathrm{f}'(x)]^2 \, [\mathrm{f}(x)]^n \mathrm{d}x \,, \qquad (*)$$

where n is a positive integer. Show that, if $\mathrm{f}(x)$ satisfies $\mathrm{f}''(x) = k\mathrm{f}(x)\mathrm{f}'(x)$ for some constant k, then $(*)$ can be integrated to obtain an expression for I in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, k and n.

(i) Verify your result in the case $\mathrm{f}(x) = \tan x$. Hence find

$$\int \frac{\sin^4 x}{\cos^8 x} \, \mathrm{d}x \,.$$

(ii) Find

$$\int \sec^2 x \, (\sec x + \tan x)^6 \, \mathrm{d}x \,.$$

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