STEP Past Papers by Topic

STEP Topic – Number theory

STEP III Specimen Question 10 (Pure)

10 **(i)** Show that every odd square leaves remainder 1 when divided by 8, and that every even square leaves remainder 0 or 4. Deduce that a number of the form $8n + 7$, where n is a positive integer, cannot be expressed as a sum of three squares.

(ii) Prove that 17 divides $2^{3n+1} + 3(5^{2n+1})$ for all integers $n \geqslant 0$.

STEP II 1997 Question 1 (Pure)

1 Find the sum of those numbers between 1000 and 6000 every one of whose digits is one of the numbers 0, 2, 5 or 7, giving your answer as a product of primes.

STEP II 1998 Question 1 (Pure)

1 Show that, if n is an integer such that

$$(n-3)^3 + n^3 = (n+3)^3, \qquad (*)$$

then n is even and n^2 is a factor of 54. Deduce that there is no integer n which satisfies the equation $(*)$.

Show that, if n is an integer such that

$$(n-6)^3 + n^3 = (n+6)^3, \qquad (**)$$

then n is even. Deduce that there is no integer n which satisfies the equation $(**)$.

STEP I 2007 Question 1 (Pure)

1 A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits $0, 1, 2, 3$ and 4.

(ii) Show that $\frac{1}{6}k\,(k+1)\,(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\displaystyle\sum_{r=0}^{n} r^2 \equiv \frac{1}{6}n\,(n+1)\,(2n+1)\,.$

STEP I 2006 Question 1 (Pure)

1 Find the integer, n, that satisfies $n^2 < 33\,127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33\,127$ is a perfect square. Hence express $33\,127$ in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of $33\,127$, show that there are exactly two values of m for which $(n+m)^2 - 33\,127$ is a perfect square, and find the other value.

STEP III 1996 Question 4 (Pure)

4 Find the integers k satisfying the inequality $k \leqslant 2(k-2)$.

Given that N is a strictly positive integer consider the problem of finding strictly positive integers whose sum is N and whose product is as large as possible. Call this largest possible product $P(N)$. Show that $P(5) = 2\times3$, $P(6) = 3^2$, $P(7) = 2^2\times3$, $P(8) = 2\times3^2$ and $P(9) = 3^3$.

Find $P(1000)$ explaining your reasoning carefully.

STEP II 2005 Question 2 (Pure)

2 For any positive integer N, the function $\mathrm{f}(N)$ is defined by

$$\mathrm{f}(N) = N\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\cdots\Big(1-\frac{1}{p_k}\Big)$$

where $p_1, p_2, \ldots, p_k$ are the only prime numbers that are factors of N.
Thus $\mathrm{f}(80) = 80(1-\frac{1}{2})(1-\frac{1}{5})$.

(i) **(a)** Evaluate $\mathrm{f}(12)$ and $\mathrm{f}(180)$.

(b) Show that $\mathrm{f}(N)$ is an integer for all N.

(ii) Prove, or disprove by means of a counterexample, each of the following:

(a) $\mathrm{f}(m)\mathrm{f}(n) = \mathrm{f}(mn)$;

(b) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ if p and q are distinct prime numbers;

(c) $\mathrm{f}(p)\mathrm{f}(q) = \mathrm{f}(pq)$ only if p and q are distinct prime numbers.

(iii) Find a positive integer m and a prime number p such that $\mathrm{f}(p^m) = 146410$.

STEP I 2003 Question 7 (Pure)

7 Let k be an integer satisfying $0 \leqslant k \leqslant 9$. Show that $0 \leqslant 10k - k^2 \leqslant 25$ and that $k(k-1)(k+1)$ is divisible by 3.

For each 3-digit number N, where $N \geqslant 100$, let S be the sum of the hundreds digit, the square of the tens digit and the cube of the units digit. Find the numbers N such that $S = N$.

[Hint: write $N = 100a + 10b + c$ where a, b and c are the digits of N.]

STEP I 2004 Question 5 (Pure)

5 The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned} A &: \; 1,\ 6,\ 11,\ 16,\ \ldots \\ B &: \; 2,\ 7,\ 12,\ 17,\ \ldots \\ C &: \; 3,\ 8,\ 13,\ 18,\ \ldots \\ D &: \; 4,\ 9,\ 14,\ 19,\ \ldots \\ E &: \; 5, 10,\ 15,\ 20,\ \ldots \end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723\,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974\,.$$

STEP I 2005 Question 1 (Pure)

1 47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

STEP I 2010 Question 8 (Pure)

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

STEP II 1996 Question 6 (Pure)

6 A *proper factor* of a positive integer N is an integer M, with $M \neq 1$ and $M \neq N$, which divides N without remainder. Show that 12 has 4 proper factors and 16 has 3.

Suppose that N has the prime factorisation

$$N = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r},$$

where $p_1, p_2, \ldots, p_r$ are distinct primes and $m_1, m_2, \ldots, m_r$ are positive integers. How many proper factors does N have and why?

Find:

(i) the smallest positive integer which has precisely 12 proper factors;

(ii) the smallest positive integer which has at least 12 proper factors.

STEP I 1999 Question 1 (Pure)

1 How many integers greater than or equal to zero and less than a million are not divisible by 2 or 5? What is the average value of these integers?

How many integers greater than or equal to zero and less than 4179 are not divisible by 3 or 7? What is the average value of these integers?

STEP I 1996 Question 3 (Pure)

3 Let n be a positive integer.

(i) Factorise $n^5 - n^3$, and show that it is divisible by 24.

(ii) Prove that $2^{2n} - 1$ is divisible by 3.

(iii) If $n - 1$ is divisible by 3, show that $n^3 - 1$ is divisible by 9.

STEP I 2011 Question 8 (Pure)

8 **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n + 1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n + 1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

STEP III 2015 Question 5 (Pure)

5 **(i)** In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set S to be the set of positive integers with the following property:

$$n \text{ is in } S \text{ if and only if } n\sqrt{2} \text{ is an integer.}$$

3. Show that the set S contains at least one positive integer.
4. Define the integer k to be the smallest positive integer in S.
5. Show that $(\sqrt{2}-1)k$ is in S.
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

(ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.

STEP I 2015 Question 8 (Pure)

8 Show that:

(i) $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$;

(ii) if N is a positive integer, m is a non-negative integer and k is a positive odd integer, then $(N-m)^k + m^k$ is divisible by N.

Let $S = 1^k + 2^k + 3^k + \cdots + n^k$, where k is a positive odd integer. Show that if n is odd then S is divisible by n and that if n is even then S is divisible by $\frac{1}{2}n$.

Show further that S is divisible by $1 + 2 + 3 + \cdots + n$.

STEP I 2014 Question 1 (Pure)

1 *All numbers referred to in this question are non-negative integers.*

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

STEP III 2011 Question 2 (Pure)

2 The polynomial $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,,$$

where $n \geqslant 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $\mathrm{f}(x) = 0$ has a rational root p/q, where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}\mathrm{f}(p/q)$, find the value of q and deduce that any rational root of the equation $\mathrm{f}(x) = 0$ must be an integer.

(i) Show that the nth root of 2 is irrational for $n \geqslant 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geqslant 2$.

STEP II 1994 Question 1 (Pure)

1 In this question we consider only positive, non-zero integers written out in the usual (decimal) way. We say, for example, that 207 ends in 7 and that 5310 ends in 1 followed by 0. Show that, if n does not end in 5 or an even number, then there exists m such that $n \times m$ ends in 1.

Show that, given any n, we can find m such that $n \times m$ ends either in 1 or in 1 followed by one or more zeros.

Show that, given any n which ends in 1 or in 1 followed by one or more zeros, we can find m such that $n \times m$ contains all the digits $0, 1, 2, \ldots, 9$.

STEP II 2013 Question 7 (Pure)

7 **(i)** Write down a solution of the equation

$$x^2 - 2y^2 = 1\,, \qquad (*)$$

for which x and y are non-negative integers.

Show that, if $x = p$, $y = q$ is a solution of $(*)$, then so also is $x = 3p + 4q$, $y = 2p + 3q$. Hence find two solutions of $(*)$ for which x is a positive odd integer and y is a positive even integer.

(ii) Show that, if x is an odd integer and y is an even integer, $(*)$ can be written in the form

$$n^2 = \tfrac{1}{2}m(m+1)\,,$$

where m and n are integers.

(iii) The positive integers a, b and c satisfy

$$b^3 = c^4 - a^2\,,$$

where b is a prime number. Express a and c^2 in terms of b in the two cases that arise.

Find a solution of $a^2 + b^3 = c^4$, where a, b and c are positive integers but b is not prime.

STEP I 2009 Question 1 (Pure)

1 A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

STEP II 2011 Question 2 (Pure)

2 Write down the cubes of the integers 1, 2, . . . , 10 .

The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3\,, \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2\,.$$

Deduce that $(4z^3 - k^2)/3$ is a perfect square and that $\frac{1}{4}k^2 \leqslant z^3 < k^2$.

Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

STEP III 2013 Question 5 (Pure)

5 In this question, you may assume that, if a, b and c are positive integers such that a and b are coprime and a divides bc, then a divides c. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers p, q, n and N such that p and q are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer k and deduce the value of q.

Hence prove that, for any positive integers n and N, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers a, b, c and d such that a and b are coprime and c and d are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if p is a prime factor of d, then p is also a prime factor of b.

If p^m and p^n are the highest powers of the prime number p that divide d and b, respectively, express b in terms of a, m and n and hence show that $p^n \leqslant n$. Deduce the value of b. (You may assume that if $x > 0$ and $y \geqslant 2$ then $y^x > x$.)

Hence prove that, if r is a positive rational number such that r^r is rational, then r is a positive integer.

STEP I 2008 Question 1 (Pure)

1 What does it mean to say that a number x is *irrational*?

Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p + q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p + q$ is irrational.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi + \mathrm{e}$, $\pi - \mathrm{e}$, $\pi^2 - \mathrm{e}^2$, $\pi^2 + \mathrm{e}^2$ is rational.
