STEP Past Papers by Topic

STEP Topic – Poisson distribution

STEP II 2003 Question 13 (Mechanics)

13 The random variable X takes the values $k = 1, 2, 3, \dots$, and has probability distribution

$$\mathrm{P}(X = k) = A\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = (1 - \mathrm{e}^{-\lambda})^{-1}$. Find the mean μ in terms of λ and show that

$$\mathrm{Var}\,(X) = \mu(1 - \mu + \lambda)\,.$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

STEP II 2005 Question 13 (Mechanics)

13 The number of printing errors on any page of a large book of N pages is modelled by a Poisson variate with parameter λ and is statistically independent of the number of printing errors on any other page. The number of pages in a random sample of n pages (where n is much smaller than N and $n \geqslant 2$) which contain fewer than two errors is denoted by Y. Show that $\mathrm{P}(Y = k) = \binom{n}{k} p^k q^{n-k}$ where $p = (1+\lambda)e^{-\lambda}$ and $q = 1 - p$.

Show also that, if λ is sufficiently small,

(i) $q \approx \frac{1}{2}\lambda^2$;

(ii) the largest value of n for which $\mathrm{P}(Y = n) \geqslant 1 - \lambda$ is approximately $2/\lambda$;

(iii) $\mathrm{P}(Y > 1 \mid Y > 0) \approx 1 - n(\lambda^2/2)^{n-1}$.

STEP III 2018 Question 13 (Probability and Statistics)

13 The random variable X takes only non-negative integer values and has probability generating function $\mathrm{G}(t)$. Show that

$$\mathrm{P}(X = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6 \;\ldots) = \frac{1}{2}\big(\mathrm{G}(1) + \mathrm{G}(-1)\big).$$

You are now given that X has a Poisson distribution with mean λ. Show that

$$\mathrm{G}(t) = \mathrm{e}^{-\lambda(1-t)}\,.$$

(i) The random variable Y is defined by

$$\mathrm{P}(Y = r) = \begin{cases} k\mathrm{P}(X = r) & \text{if } r = 0,\ 2,\ 4,\ 6,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where k is an appropriate constant.

Show that the probability generating function of Y is $\dfrac{\cosh \lambda t}{\cosh \lambda}$.

Deduce that $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

(ii) The random variable Z is defined by

$$\mathrm{P}(Z = r) = \begin{cases} c\mathrm{P}(X = r) & \text{if } r = 0,\ 4,\ 8,\ 12,\ \ldots\ , \\ 0 & \text{otherwise,} \end{cases}$$

where c is an appropriate constant.

Is $\mathrm{E}(Z) < \lambda$ for all positive values of λ?

STEP I 2015 Question 12 (Probability and Statistics)

12 The number X of casualties arriving at a hospital each day follows a Poisson distribution with mean 8; that is,

$$P(X = n) = \frac{e^{-8}8^n}{n!}, \quad n = 0, 1, 2, \ldots .$$

Casualties require surgery with probability $\frac{1}{4}$. The number of casualties arriving on any given day is independent of the number arriving on any other day and the casualties require surgery independently of one another.

(i) What is the probability that, on a day when exactly n casualties arrive, exactly r of them require surgery?

(ii) Prove (algebraically) that the number requiring surgery each day also follows a Poisson distribution, and state its mean.

(iii) Given that in a particular randomly chosen week a total of 12 casualties require surgery on Monday and Tuesday, what is the probability that 8 casualties require surgery on Monday? You should give your answer as a fraction in its lowest terms.

STEP II 1995 Question 13 (Mechanics)

13 Fly By Night Airlines run jumbo jets which seat N passengers. From long experience they know that a very small proportion ϵ of their passengers fail to turn up. They decide to sell $N+k$ tickets for each flight. If k is very small compared with N explain why they might expect

$$\mathrm{P}(r \text{ passengers fail to turn up}) = \frac{\lambda^r}{r!}\mathrm{e}^{-\lambda}$$

approximately, with $\lambda = N\epsilon$. For the rest of the question you may assume that the formula holds exactly.

Each ticket sold represents £A profit, but the airline must pay each passenger that it cannot fly £B where $B > A > 0$. Explain why, if r passengers fail to turn up, its profit, in pounds, is

$$A(N+k) - B\max(0, k-r),$$

where $\max(0, k-r)$ is the larger of 0 and $k-r$. Write down the expected profit u_k when $k = 0, 1, 2$ and 3. Find $v_k = u_{k+1} - u_k$ for general k and show that $v_k > v_{k+1}$. Show also that

$$v_k \to A - B$$

as $k \to \infty$.

Advise Fly By Night on how to choose k to maximise its expected profit u_k.

STEP I 2010 Question 13 (Probability and Statistics)

13 The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

STEP I 1995 Question 13 (Probability and Statistics)

13 A scientist is checking a sequence of microscope slides for cancerous cells, marking each cancerous cell that she detects with a red dye. The number of cancerous cells on a slide is random and has a Poisson distribution with mean μ. The probability that the scientist spots any one cancerous cell is p, and is independent of the probability that she spots any other one.

(i) Show that the number of cancerous cells which she marks on a single slide has a Poisson distribution of mean $p\mu$.

(ii) Show that the probability Q that the second cancerous cell which she marks is on the kth slide is given by

$$Q = \mathrm{e}^{-\mu p(k-1)}\left\{(1 + k\mu p)(1 - \mathrm{e}^{-\mu p}) - \mu p\right\}.$$

STEP II 2013 Question 12 (Probability and Statistics)

12 The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1, 3, 5, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2, 4, 6, 8, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots .$$

(ii) Show that

$$\mathrm{Var}\,(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\mathrm{Var}\,(Y)$. Are there any non-zero values of λ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X+Y)$?

STEP III 2003 Question 12 (Probability and Statistics)

12 Brief interruptions to my work occur on average every ten minutes and the number of interruptions in any given time period has a Poisson distribution. Given that an interruption has just occurred, find the probability that I will have less than t minutes to work before the next interruption. If the random variable T is the time I have to work before the next interruption, find the probability density function of T.

I need an uninterrupted half hour to finish an important paper. Show that the expected number of interruptions before my first uninterrupted period of half an hour or more is $e^3 - 1$. Find also the expected length of time between interruptions that are less than half an hour apart. Hence write down the expected wait before my first uninterrupted period of half an hour or more.

STEP I 2001 Question 13 (Probability and Statistics)

13 Four students, one of whom is a mathematician, take turns at washing up over a long period of time. The number of plates broken by any student in this time obeys a Poisson distribution, the probability of any given student breaking n plates being $e^{-\lambda}\lambda^n/n!$ for some fixed constant λ, independent of the number of breakages by other students. Given that five plates are broken, find the probability that three or more were broken by the mathematician.

STEP I 2007 Question 14 (Probability and Statistics)

14 The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x+1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

It is known that $\mathrm{P}(X \geqslant 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathrm{P}\,(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathrm{P}\,(X = 1 \,|\, X \leqslant 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).

STEP I 1994 Question 14 (Probability and Statistics)

14 Each of my n students has to hand in an essay to me. Let T_i be the time at which the ith essay is handed in and suppose that $T_1, T_2, \ldots, T_n$ are independent, each with probability density function $\lambda e^{-\lambda t}$ $(t \geqslant 0)$. Let T be the time I receive the first essay to be handed in and let U be the time I receive the last one.

(i) Find the mean and variance of T_i.

(ii) Show that $P(U \leqslant u) = (1 - e^{-\lambda u})^n$ for $u \geqslant 0$, and hence find the probability density function of U.

(iii) Obtain $P(T > t)$, and hence find the probability density function of T.

(iv) Write down the mean and variance of T.

STEP II 1998 Question 14 (Mechanics)

14 The staff of Catastrophe College are paid a salary of A pounds per year. With a Teaching Assessment Exercise impending it is decided to try to lower the student failure rate by offering each lecturer an alternative salary of $B/(1+X)$ pounds, where X is the number of his or her students who fail the end of year examination. Dr Doom has N students, each with independent probability p of failure. Show that she should accept the new salary scheme if

$$A(N+1)p < B(1-(1-p)^{N+1}).$$

Under what circumstances could X, for Dr Doom, be modelled by a Poisson random variable? What would Dr Doom's expected salary be under this model?

STEP II 1997 Question 14 (Mechanics)

14 Traffic enters a tunnel which is 9600 metres long, and in which overtaking is impossible. The number of vehicles which enter in any given time is governed by the Poisson distribution with mean 6 cars per minute. All vehicles travel at a constant speed until forced to slow down on catching up with a slower vehicle ahead. I enter the tunnel travelling at 30 $\mathrm{m\,s^{-1}}$ and all the other traffic is travelling at 32 $\mathrm{m\,s^{-1}}$. What is the expected number of vehicles in the queue behind me when I leave the tunnel?

Assuming again that I travel at 30 $\mathrm{m\,s^{-1}}$, but that all the other vehicles are independently equally likely to be travelling at 30 $\mathrm{m\,s^{-1}}$ or 32 $\mathrm{m\,s^{-1}}$, find the probability that exactly two vehicles enter the tunnel within 20 seconds of my doing so and catch me up before I leave it. Find also the probability that there are exactly two vehicles queuing behind me when I leave the tunnel.

[Ignore the lengths of the vehicles.]

STEP III 1999 Question 14 (Probability and Statistics)

14 In the basic version of Horizons (H1) the player has a maximum of n turns, where $n \geqslant 1$. At each turn, she has a probability p of success, where $0 < p < 1$. If her first success is at the rth turn, where $1 \leqslant r \leqslant n$, she collects r pounds and then withdraws from the game. Otherwise, her winnings are nil. Show that in H1, her expected winnings are

$$p^{-1}\left[1 + nq^{n+1} - (n+1)q^n\right] \quad \text{pounds,}$$

where $q = 1 - p$.

The rules of H2 are the same as those of H1, except that n is randomly selected from a Poisson distribution with parameter λ. If $n = 0$ her winnings are nil. Otherwise she plays H1 with the selected n. Show that in H2, her expected winnings are

$$\frac{1}{p}\left(1 - \mathrm{e}^{-\lambda p}\right) - \lambda q \mathrm{e}^{-\lambda p} \quad \text{pounds.}$$

STEP I 2003 Question 14 (Probability and Statistics)

14 Jane goes out with any of her friends who call, except that she never goes out with more than two friends in a day. The number of her friends who call on a given day follows a Poisson distribution with parameter 2. Show that the average number of friends she sees in a day is $2 - 4\mathrm{e}^{-2}$.

Now Jane has a new friend who calls on any given day with probability p. Her old friends call as before, independently of the new friend. She never goes out with more than two friends in a day. Find the average number of friends she now sees in a day.

STEP I 2002 Question 14 (Probability and Statistics)

14 In order to get money from a cash dispenser I have to punch in an identification number. I have forgotten my identification number, but I do know that it is equally likely to be any one of the integers $1, 2, \ldots, n$. I plan to punch in integers in order until I get the right one. I can do this at the rate of r integers per minute. As soon as I punch in the first wrong number, the police will be alerted. The probability that they will arrive within a time t minutes is $1 - \mathrm{e}^{-\lambda t}$, where λ is a positive constant. If I follow my plan, show that the probability of the police arriving before I get my money is

$$\sum_{k=1}^{n} \frac{1 - \mathrm{e}^{-\lambda(k-1)/r}}{n} \,.$$

Simplify the sum.

On past experience, I know that I will be so flustered that I will just punch in possible integers at random, without noticing which I have already tried. Show that the probability of the police arriving before I get my money is

$$1 - \frac{1}{n - (n-1)\mathrm{e}^{-\lambda/r}} \,.$$

STEP III 1992 Question 16 (Probability and Statistics)

16 The probability that there are exactly n misprints in an issue of a newspaper is $e^{-\lambda}\lambda^n/n!$ where λ is a positive constant. The probability that I spot a particular misprint is p, independent of what happens for other misprints, and $0 < p < 1$.

(i) If there are exactly $m+n$ misprints, what is the probability that I spot exactly m of them?

(ii) Show that, if I spot exactly m misprints, the probability that I have failed to spot exactly n misprints is

$$\frac{(1-p)^n\lambda^n}{n!}e^{-(1-p)\lambda}.$$