STEP Past Papers by Topic

STEP Topic - Projectile motion

STEP I 1992 Question 10 (Mechanics)

10 A projectile of mass m is fired horizontally from a toy cannon of mass M which slides freely on a horizontal floor. The length of the barrel is l and the force exerted on the projectile has the constant value P for so long as the projectile remains in the barrel. Initially the cannon is at rest. Show that the projectile emerges from the barrel at a speed relative to the ground of

$$\sqrt{\frac{2PMl}{m(M+m)}}.$$

STEP II 2011 Question 10 (Pure)

10 A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ.

A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\frac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

$$(\sqrt{2}-1)\sqrt{H/g}\,.$$

When the string becomes taut, R leaves the plane, the string remaining taut. Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ.

Given that $D = H$, find the value of $\tan\theta$.

STEP I 2017 Question 9 (Mechanics)

9 A particle is projected at speed u from a point O on a horizontal plane. It passes through a fixed point P which is at a horizontal distance d from O and at a height $d\tan\beta$ above the plane, where $d > 0$ and β is an acute angle. The angle of projection α is chosen so that u is as small as possible.

(i) Show that $u^2 = gd\tan\alpha$ and $2\alpha = \beta + 90°$.

(ii) At what angle to the horizontal is the particle travelling when it passes through P? Express your answer in terms of α in its simplest form.

STEP II 1994 Question 11 (Mechanics)

11 As part of a firework display a shell is fired vertically upwards with velocity v from a point on a level stretch of ground. When it reaches the top of its trajectory an explosion it splits into two equal fragments each travelling at speed u but (since momentum is conserved) in exactly opposite (not necessarily horizontal) directions. Show, neglecting air resistance, that the greatest possible distance between the points where the two fragments hit the ground is $2uv/g$ if $u \leqslant v$ and $(u^2 + v^2)/g$ if $v \leqslant u$.

STEP II 1991 Question 11 (Mechanics)

11 The Ruritanian army is supplied with shells which may explode at any time in flight but not before the shell reaches its maximum height. The effect of the explosion on any observer depends only on the distance between the exploding shell and the observer (and decreases with distance). Ruritanian guns fire the shells with fixed muzzle speed, and it is the policy of the gunners to fire the shell at an angle of elevation which minimises the possible damages to themselves (assuming the ground is level) - i.e. they aim so that the point on the descending trajectory that is nearest to them is as far away as possible. With that intention, they choose the angle of elevation that minimises the damage to themselves if the shell explodes at its maximum height. What angle do they choose?

Does the shell then get any nearer to the gunners during its descent?

STEP II Specimen Question 11 (Mechanics)

11 Two points A and B are at a distance a apart on a horizontal plane. A particle of mass m is projected from A with speed V, at an angle of elevation of $45°$ to the line AB. Another particle, also of mass m, is projected from B with speed U at an angle of elevation of $30°$ to the line BA so that the two particles collide at the instant when each particle is at the highest point of its trajectory.

Show that $U^2 = 2V^2$ and that

$$a = \frac{V^2}{2g}(1 + \sqrt{3}).$$

At impact the two particles coalesce. When the combined particle strikes the horizontal plane the velocity of the particle is inclined at an angle ϕ to the horizontal. Show that $\tan\phi = 1+\sqrt{3}$.

STEP III 1991 Question 12 (Mechanics)

12 A smooth tube whose axis is horizontal has an elliptic cross-section in the form of the curve with parametric equations

$$x = a\cos\theta \qquad y = b\sin\theta$$

where the x-axis is horizontal and the y-axis is vertically upwards. A particle moves freely under gravity on the inside of the tube in the plane of this cross-section. By first finding $\ddot{x}$ and $\ddot{y}$, or otherwise, show that the acceleration along the inward normal at the point with parameter θ is

$$\frac{ab\dot{\theta}^2}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}.$$

The particle is projected along the surface in the vertical cross-section plane, with speed $2\sqrt{bg}$, from the lowest point. Given that $2a = 3b$, show that it will leave the surface at the point with parameter θ where

$$5\sin^3\theta + 12\sin\theta - 8 = 0.$$

STEP I 1989 Question 11 (Mechanics)

11 A shot-putter projects a shot at an angle θ above the horizontal, releasing it at height h above the level ground, with speed v. Show that the distance R travelled horizontally by the shot from its point of release until it strikes the ground is given by

$$R = \frac{v^2}{2g}\sin 2\theta\left(1+\sqrt{1+\frac{2hg}{v^2\sin^2\theta}}\right).$$

The shot-putter's style is such that currently $\theta = 45°$. Determine (with justification) whether a small decrease in θ will increase R.

[Air resistance may be neglected.]

STEP III 1990 Question 13 (Mechanics)

13 A particle P is projected, from the lowest point, along the smooth inside surface of a fixed sphere with centre O. It leaves the surface when OP makes an angle θ with the upward vertical. Find the smallest angle that must be exceeded by θ to ensure that P will strike the surface below the level of O.

[You may find it helpful to find the time at which the particle strikes the sphere.]

STEP II 1992 Question 11 (Mechanics)

11 I am standing next to an ice-cream van at a distance d from the top of a vertical cliff of height h. It is not safe for me to go any nearer to the top of the cliff. My niece Padma is on the broad level beach at the foot of the cliff. I have just discovered that I have left my wallet with her, so I cannot buy her an ice-cream unless she can throw the wallet up to me. She can throw it at speed V, at any angle she chooses and from anywhere on the beach. Air resistance is negligible; so is Padma's height compared to that of the cliff. Show that she can throw the wallet to me if and only if

$$V^2 \geqslant g(2h + d).$$

STEP I 1993 Question 12 (Mechanics)

12 In a clay pigeon shoot the target is launched vertically from ground level with speed v. At a time T later the competitor fires a rifle inclined at angle α to the horizontal. The competitor is also at ground level and is a distance l from the launcher. The speed of the bullet leaving the rifle is u. Show that, if the competitor scores a hit, then

$$l \sin\alpha - \left(vT - \tfrac{1}{2}gT^2\right)\cos\alpha = \frac{v - gT}{u} l.$$

Suppose now that $T = 0$. Show that if the competitor can hit the target before it hits the ground then $v < u$ and

$$\frac{2v\sqrt{u^2 - v^2}}{g} > l.$$

STEP I 1990 Question 11 (Mechanics)

11 A shell of mass m is fired at elevation $\pi/3$ and speed v. Superman, of mass $2m$, catches the shell at the top of its flight, by gliding up behind it in the same horizontal direction with speed $3v$. As soon as Superman catches the shell, he instantaneously clasps it in his cloak, and immediately pushes it vertically downwards, without further changing its horizontal component of velocity, but giving it a downward vertical component of velocity of magnitude $3v/2$. Calculate the total time of flight of the shell in terms of v and g. Calculate also, to the nearest degree, the angle Superman's flight trajectory initially makes with the horizontal after releasing the shell, as he soars upwards like a bird.

[Superman and the shell may be regarded as particles.]

STEP I 1988 Question 10 (Mechanics)

10 A sniper at the top of a tree of height h is hit by a bullet fired from the undergrowth covering the horizontal ground below. The position and elevation of the gun which fired the shot are unknown, but it is known that the bullet left the gun with speed v. Show that it must have been fired from a point within a circle centred on the base of the tree and of radius $(v/g)\sqrt{v^2 - 2gh}$.

[Neglect air resistance.]

STEP I 1997 Question 11 (Mechanics)

11 A particle of unit mass is projected vertically upwards in a medium whose resistance is k times the square of the velocity of the particle. If the initial velocity is u, prove that the velocity v after rising through a distance s satisfies

$$v^2 = u^2 e^{-2ks} + \frac{g}{k}(e^{-2ks} - 1). \qquad (*)$$

Find an expression for the maximum height of the particle above the point of projection.

Does equation $(*)$ still hold on the downward path? Justify your answer.

STEP III 2003 Question 11 (Mechanics)

11 Point B is a distance d due south of point A on a horizontal plane. Particle P is at rest at B at $t = 0$, when it begins to move with constant acceleration a in a straight line with fixed bearing β. Particle Q is projected from point A at $t = 0$ and moves in a straight line with constant speed v. Show that if the direction of projection of Q can be chosen so that Q strikes P, then

$$v^2 \geqslant ad\,(1 - \cos\beta) \ .$$

Show further that if $v^2 > ad(1 - \cos\beta)$ then the direction of projection of Q can be chosen so that Q strikes P before P has moved a distance d.

STEP II 2001 Question 11 (Mechanics)

11 A two-stage missile is projected from a point A on the ground with horizontal and vertical velocity components u and v, respectively. When it reaches the highest point of its trajectory an internal explosion causes it to break up into two fragments. Immediately after this explosion one of these fragments, P, begins to move vertically upwards with speed v_e, but retains the previous horizontal velocity. Show that P will hit the ground at a distance R from A given by

$$\frac{gR}{u} = v + v_e + \sqrt{v_e^2 + v^2}\,.$$

It is required that the range R should be greater than a certain distance D (where $D > 2uv/g$). Show that this requirement is satisfied if

$$v_e > \frac{gD}{2u}\left(\frac{gD - 2uv}{gD - uv}\right).$$

[*The effect of air resistance is to be neglected.*]

STEP II 2012 Question 9 (Mechanics)

9 A tennis ball is projected from a height of $2h$ above horizontal ground with speed u and at an angle of α below the horizontal. It travels in a plane perpendicular to a vertical net of height h which is a horizontal distance of a from the point of projection. Given that the ball passes over the net, show that

$$\frac{1}{u^2} < \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}\,.$$

The ball lands before it has travelled a horizontal distance of b from the point of projection. Show that

$$\sqrt{u^2\sin^2\alpha + 4gh} < \frac{bg}{u\cos\alpha} + u\sin\alpha\,.$$

Hence show that

$$\tan\alpha < \frac{h(b^2 - 2a^2)}{ab(b-a)}\,.$$

STEP II 1995 Question 11 (Mechanics)

11 Two identical particles of unit mass move under gravity in a medium for which the magnitude of the retarding force on a particle is k times its speed. The first particle is allowed to fall from rest at a point A whilst, at the same time, the second is projected upwards with speed u from a point B a positive distance d vertically above A. Find their distance apart after a time t and show that this distance tends to the value

$$d + \frac{u}{k}$$

as $t \to \infty$.

STEP II 1999 Question 11 (Mechanics)

11 An automated mobile dummy target for gunnery practice is moving anti-clockwise around the circumference of a large circle of radius R in a horizontal plane at a constant angular speed ω. A shell is fired from O, the centre of this circle, with initial speed V and angle of elevation α. Show that if $V^2 < gR$, then no matter what the value of α, or what vertical plane the shell is fired in, the shell cannot hit the target.

Assume now that $V^2 > gR$ and that the shell hits the target, and let β be the angle through which the target rotates between the time at which the shell is fired and the time of impact. Show that β satisfies the equation

$$g^2\beta^4 - 4\omega^2V^2\beta^2 + 4R^2\omega^4 = 0.$$

Deduce that there are exactly two possible values of β.

Let β_1 and β_2 be the possible values of β and let P_1 and P_2 be the corresponding points of impact. By considering the quantities $(\beta_1^2 + \beta_2^2)$ and $\beta_1^2\beta_2^2$, or otherwise, show that the linear distance between P_1 and P_2 is

$$2R\sin\left(\frac{\omega}{g}\sqrt{V^2 - Rg}\right).$$

STEP I 2007 Question 11 (Mechanics)

11 A smooth, straight, narrow tube of length L is fixed at an angle of 30° to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motion until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the tube. Show that D satisfies the equation

$$4gD^2 - 2\sqrt{3}\left(u^2 - Lg\right)D - 3L\left(u^2 - gL\right) = 0$$

and hence that

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\,(u^2 - gL)}.$$

The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

STEP I 2006 Question 10 (Mechanics)

10 A particle P is projected in the x-y plane, where the y-axis is vertical and the x-axis is horizontal. The particle is projected with speed V from the origin at an angle of $45°$ above the positive x-axis. Determine the equation of the trajectory of P.

The point of projection (the origin) is on the floor of a barn. The roof of the barn is given by the equation $y = x\tan\alpha + b$, where $b > 0$ and α is an acute angle. Show that, if the particle just touches the roof, then $V(-1+\tan\alpha) = -2\sqrt{bg}$; you should justify the choice of the negative root. If this condition is satisfied, find, in terms of α, V and g, the time after projection at which touching takes place.

A particle Q can slide along a smooth rail fixed, in the x-y plane, to the under-side of the roof. It is projected from the point $(0, b)$ with speed U at the same time as P is projected from the origin. Given that the particles just touch in the course of their motions, show that

$$2\sqrt{2}\,U\cos\alpha = V\big(2 + \sin\alpha\cos\alpha - \sin^2\alpha\big).$$

STEP I 2003 Question 9 (Mechanics)

9 A particle is projected with speed V at an angle θ above the horizontal. The particle passes through the point P which is a horizontal distance d and a vertical distance h from the point of projection. Show that

$$T^2 - 2kT + \frac{2kh}{d} + 1 = 0\,,$$

where $T = \tan\theta$ and $k = \dfrac{V^2}{gd}$.

Show that, if $kd > h + \sqrt{h^2 + d^2}$, there are two distinct possible angles of projection.

Let these two angles be α and β. Show that $\alpha + \beta = \pi - \arctan(d/h)$.

STEP II 1997 Question 11 (Mechanics)

11 A tennis player serves from height H above horizontal ground, hitting the ball downwards with speed v at an angle α below the horizontal. The ball just clears the net of height h at horizontal distance a from the server and hits the ground a further horizontal distance b beyond the net. Show that

$$v^2 = \frac{g(a+b)^2(1+\tan^2\alpha)}{2[H-(a+b)\tan\alpha]}$$

and

$$\tan\alpha = \frac{2a+b}{a(a+b)}H - \frac{a+b}{ab}h\,.$$

By considering the signs of v^2 and $\tan\alpha$, find upper and lower bounds on H for such a serve to be possible.

STEP I 2012 Question 9 (Mechanics)

9 A tall shot-putter projects a small shot from a point 2.5 m above the ground, which is horizontal. The speed of projection is $10\,\text{m}\,\text{s}^{-1}$ and the angle of projection is θ above the horizontal. Taking the acceleration due to gravity to be $10\,\text{m}\,\text{s}^{-2}$, show that the time, in seconds, that elapses before the shot hits the ground is

$$\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right),$$

where $c = \cos 2\theta$.

Find an expression for the range in terms of c and show that it is greatest when $c = \frac{1}{5}$.

Show that the extra distance attained by projecting the shot at this angle rather than at an angle of $45°$ is $5(\sqrt{6}-\sqrt{2}-1)$ m.

STEP II 2010 Question 9 (Mechanics)

9 Two points A and B lie on horizontal ground. A particle P_1 is projected from A towards B at an acute angle of elevation α and simultaneously a particle P_2 is projected from B towards A at an acute angle of elevation β. Given that the two particles collide in the air a horizontal distance b from B, and that the collision occurs after P_1 has attained its maximum height h, show that

$$2h\cot\beta < b < 4h\cot\beta$$

and

$$2h\cot\alpha < a < 4h\cot\alpha\,,$$

where a is the horizontal distance from A to the point of collision.

STEP I 1998 Question 10 (Mechanics)

10 A shell explodes on the surface of horizontal ground. Earth is scattered in all directions with varying velocities. Show that particles of earth with initial speed v landing a distance r from the centre of explosion will do so at times t given by

$$\tfrac{1}{2}g^2t^2 = v^2 \pm \sqrt{(v^4 - g^2r^2)}.$$

Find an expression in terms of v, r and g for the greatest height reached by such particles.

STEP I 2000 Question 9 (Mechanics)

9 A child is playing with a toy cannon on the floor of a long railway carriage. The carriage is moving horizontally in a northerly direction with acceleration a. The child points the cannon southward at an angle θ to the horizontal and fires a toy shell which leaves the cannon at speed V. Find, in terms of a and g, the value of $\tan 2\theta$ for which the cannon has maximum range (in the carriage).

If a is small compared with g, show that the value of θ which gives the maximum range is approximately

$$\frac{\pi}{4} + \frac{a}{2g},$$

and show that the maximum range is approximately $\dfrac{V^2}{g} + \dfrac{V^2 a}{g^2}$.

STEP II 2003 Question 11 (Mechanics)

11 A particle P_1 is projected with speed V at an angle of elevation α $(> 45°)$, from a point in a horizontal plane. Find T_1, the flight time of P_1, in terms of α, V and g. Show that the time after projection at which the direction of motion of P_1 first makes an angle of $45°$ with the horizontal is $\frac{1}{2}(1 - \cot\alpha)T_1$.

A particle P_2 is projected under the same conditions. When the direction of the motion of P_2 first makes an angle of $45°$ with the horizontal, the speed of P_2 is instantaneously doubled. If T_2 is the total flight time of P_2, show that

$$\frac{2T_2}{T_1} = 1 + \cot\alpha + \sqrt{1 + 3\cot^2\alpha}\,.$$

STEP III 1995 Question 10 (Mechanics)

10 A cannon is situated at the bottom of a plane inclined at angle β to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed u. Ignoring air resistance, find the angle of firing which will maximise the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1+\sin\beta)}$$

from the cannon.

STEP III 2007 Question 10 (Mechanics)

10 A particle is projected from a point on a plane that is inclined at an angle ϕ to the horizontal. The position of the particle at time t after it is projected is (x, y), where $(0, 0)$ is the point of projection, x measures distance up the line of greatest slope and y measures perpendicular distance from the plane. Initially, the velocity of the particle is given by $(\dot{x}, \dot{y}) = (V\cos\theta, V\sin\theta)$, where $V > 0$ and $\phi + \theta < \pi/2$. Write down expressions for x and y.

The particle bounces on the plane and returns along the same path to the point of projection. Show that

$$2\tan\phi\tan\theta = 1$$

and that

$$R = \frac{V^2\cos^2\theta}{2g\sin\phi},$$

where R is the range along the plane.

Show further that

$$\frac{2V^2}{gR} = 3\sin\phi + \operatorname{cosec}\phi$$

and deduce that the largest possible value of R is $V^2/(\sqrt{3}\,g)$.

STEP II 2007 Question 11 (Mechanics)

11 *In this question take the acceleration due to gravity to be* $10\,\mathrm{m\,s^{-2}}$ *and neglect air resistance.*

The point O lies in a horizontal field. The point B lies 50 m east of O. A particle is projected from B at speed $25\,\mathrm{m\,s^{-1}}$ at an angle $\arctan\frac{1}{2}$ above the horizontal and in a direction that makes an angle $60°$ with OB; it passes to the north of O.

(i) Taking unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to O at time t seconds after the particle was projected, and show that its distance from O is

$$5(t^2 - \sqrt{5}t + 10)\,\mathrm{m}.$$

When this distance is shortest, the particle is at point P. Find the position vector of P and its horizontal bearing from O.

(ii) Show that the particle reaches its maximum height at P.

(iii) When the particle is at P, a marksman fires a bullet from O directly at P. The initial speed of the bullet is $350\,\mathrm{m\,s^{-1}}$. Ignoring the effect of gravity on the bullet show that, when it passes through P, the distance between P and the particle is approximately 3 m.

STEP II 2006 Question 11 (Mechanics)

11 A projectile of unit mass is fired in a northerly direction from a point on a horizontal plain at speed u and an angle θ above the horizontal. It lands at a point A on the plain. In flight, the projectile experiences two forces: gravity, of magnitude g; and a horizontal force of constant magnitude f due to a wind blowing from North to South. Derive an expression, in terms of u, g, f and θ for the distance OA.

(i) Determine the angle α such that, for all $\theta > \alpha$, the wind starts to blow the projectile back towards O before it lands at A.

(ii) An identical projectile, which experiences the same forces, is fired from O in a northerly direction at speed u and angle $45°$ above the horizontal and lands at a point B on the plain. Given that θ is chosen to maximise OA, show that

$$\frac{OB}{OA} = \frac{g - f}{\sqrt{g^2 + f^2} - f} \; .$$

Describe carefully the motion of the second projectile when $f = g$.

STEP I 2004 Question 9 (Mechanics)

9 A particle is projected over level ground with a speed u at an angle θ above the horizontal. Derive an expression for the greatest height of the particle in terms of u, θ and g.

A particle is projected from the floor of a horizontal tunnel of height $\frac{9}{10}d$. Point P is $\frac{1}{2}d$ metres vertically and d metres horizontally along the tunnel from the point of projection. The particle passes through point P and lands inside the tunnel without hitting the roof. Show that

$$\arctan\tfrac{3}{5} < \theta < \arctan 3\,.$$

STEP II 1998 Question 11 (Mechanics)

11 A fielder, who is perfectly placed to catch a ball struck by the batsman in a game of cricket, watches the ball in flight. Assuming that the ball is struck at the fielder's eye level and is caught just in front of her eye, show that $\frac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ is constant, where θ is the angle between the horizontal and the fielder's line of sight.

In order to catch the next ball, which is also struck towards her but at a different velocity, the fielder runs at constant speed v towards the batsman. Assuming that the ground is horizontal, show that the fielder should choose v so that $\frac{\mathrm{d}}{\mathrm{dt}}(\tan\theta)$ remains constant.

STEP I 1995 Question 9 (Mechanics)

9 A particle is projected from a point O with speed $\sqrt{2gh}$, where g is the acceleration due to gravity. Show that it is impossible, whatever the angle of projection, for the particle to reach a point above the parabola

$$x^2 = 4h(h - y),$$

where x is the horizontal distance from O and y is the vertical distance above O. State briefly the simplifying assumptions which this solution requires.

STEP I 2009 Question 9 (Mechanics)

9 Two particles P and Q are projected simultaneously from points O and D, respectively, where D is a distance d directly above O. The initial speed of P is V and its angle of projection *above* the horizontal is α. The initial speed of Q is kV, where $k > 1$, and its angle of projection *below* the horizontal is β. The particles collide at time T after projection.

Show that $\cos\alpha = k\cos\beta$ and that T satisfies the equation

$$(k^2 - 1)V^2T^2 + 2dVT\sin\alpha - d^2 = 0\,.$$

Given that the particles collide when P reaches its maximum height, find an expression for $\sin^2\alpha$ in terms of g, d, k and V, and deduce that

$$gd \leqslant (1 + k)V^2\,.$$

STEP I 2015 Question 9 (Mechanics)

9 A short-barrelled machine gun stands on horizontal ground. The gun fires bullets, from ground level, at speed u continuously from $t = 0$ to $t = \dfrac{\pi}{6\lambda}$, where λ is a positive constant, but does not fire outside this time period. During this time period, the angle of elevation α of the barrel decreases from $\frac{1}{3}\pi$ to $\frac{1}{6}\pi$ and is given at time t by

$$\alpha = \tfrac{1}{3}\pi - \lambda t\,.$$

Let $k = \dfrac{g}{2\lambda u}$. Show that, in the case $\frac{1}{2} \leqslant k \leqslant \frac{1}{2}\sqrt{3}$, the last bullet to hit the ground does so at a distance

$$\frac{2ku^2\sqrt{1-k^2}}{g}$$

from the gun.

What is the corresponding result if $k < \frac{1}{2}$?

STEP I 2014 Question 9 (Mechanics)

9 A particle of mass m is projected due east at speed U from a point on horizontal ground at an angle θ above the horizontal, where $0 < \theta < 90°$. In addition to the gravitational force mg, it experiences a horizontal force of magnitude mkg, where k is a positive constant, acting due west in the plane of motion of the particle. Determine expressions in terms of U, θ and g for the time, T_H, at which the particle reaches its greatest height and the time, T_L, at which it lands.

Let $T = U\cos\theta/(kg)$. By considering the relative magnitudes of T_H, T_L and T, or otherwise, sketch the trajectory of the particle in the cases $k\tan\theta < \frac{1}{2}$, $\frac{1}{2} < k\tan\theta < 1$, and $k\tan\theta > 1$. What happens when $k\tan\theta = 1$?

STEP I 1996 Question 11 (Mechanics)

11 A particle is projected under the influence of gravity from a point O on a level plane in such a way that, when its horizontal distance from O is c, its height is h. It then lands on the plane at a distance $c + d$ from O. Show that the angle of projection α satisfies

$$\tan\alpha = \frac{h(c+d)}{cd}$$

and that the speed of projection v satisfies

$$v^2 = \frac{g}{2}\left(\frac{cd}{h} + \frac{(c+d)^2 h}{cd}\right).$$

STEP II 2008 Question 9 (Mechanics)

9 In this question, use $g = 10\,\text{m}\,\text{s}^{-2}$.

In cricket, a fast bowler projects a ball at $40\,\text{m}\,\text{s}^{-1}$ from a point h m above the ground, which is horizontal, and at an angle α above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of 20 m from the point of projection.

(i) Determine, in terms of h, the two possible values of $\tan\alpha$.

Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.

(ii) State the range of values of h for which the bowler projects the ball below the horizontal.

(iii) In the case $h = 2.5$, give an approximate value in degrees, correct to two significant figures, for α. You need not justify the accuracy of your approximation.

[You may use the small-angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.]

STEP II 2013 Question 10 (Pure)

10 A particle is projected at an angle of elevation α (where $\alpha > 0$) from a point A on horizontal ground. At a general point in its trajectory the angle of elevation of the particle from A is θ and its direction of motion is at an angle ϕ above the horizontal (with $\phi \geqslant 0$ for the first half of the trajectory and $\phi \leqslant 0$ for the second half).

Let B denote the point on the trajectory at which $\theta = \frac{1}{2}\alpha$ and let C denote the point on the trajectory at which $\phi = -\frac{1}{2}\alpha$.

(i) Show that, at a general point on the trajectory, $2\tan\theta = \tan\alpha + \tan\phi$.

(ii) Show that, if B and C are the same point, then $\alpha = 60°$.

(iii) Given that $\alpha < 60°$, determine whether the particle reaches the point B first or the point C first.

STEP I 2008 Question 10 (Mechanics)

10 On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2 \sin^2 \alpha > 2gh$. Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha \,,$$

where $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

STEP II 2014 Question 10 (Pure)

10 A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O. Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g.

Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u.

Hence find the furthest distance from O that can be achieved by such a projectile.

STEP III 2009 Question 9 (Mechanics)

9 A particle is projected under gravity from a point P and passes through a point Q. The angles of the trajectory with the positive horizontal direction at P and at Q are θ and ϕ, respectively. The angle of elevation of Q from P is α.

(i) Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

(ii) It is given that there is a second trajectory from P to Q with the same speed of projection. The angles of this trajectory with the positive horizontal direction at P and at Q are θ' and ϕ', respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta + \theta') = -\cot\alpha$. Show also that $\theta + \theta' = \pi + \phi + \phi'$.

STEP I 2011 Question 9 (Mechanics)

9 A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

STEP II 2005 Question 10 (Pure)

10 The points A and B are 180 metres apart and lie on horizontal ground. A missile is launched from A at speed of $100\,\text{m s}^{-1}$ and at an acute angle of elevation to the line AB of $\arcsin \frac{3}{5}$. A time T seconds later, an anti-missile missile is launched from B, at speed of $200\,\text{m s}^{-1}$ and at an acute angle of elevation to the line BA of $\arcsin \frac{4}{5}$. The motion of both missiles takes place in the vertical plane containing A and B, and the missiles collide.

Taking $g = 10\,\text{m s}^{-2}$ and ignoring air resistance, find T.

[Note that $\arcsin \frac{3}{5}$ is another notation for $\sin^{-1} \frac{3}{5}$.]

STEP I 2013 Question 9 (Mechanics)

9 Two particles, A and B, are projected simultaneously towards each other from two points which are a distance d apart in a horizontal plane. Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has mass M and is projected at speed v at angle β above the horizontal. The trajectories of the two particles lie in the same vertical plane.

The particles collide directly when each is at its point of greatest height above the plane. Given that both A and B return to their starting points, and that momentum is conserved in the collision, show that

$$m \cot \alpha = M \cot \beta \,.$$

Show further that the collision occurs at a point which is a horizontal distance b from the point of projection of A where

$$b = \frac{Md}{m+M} \,,$$

and find, in terms of b and α, the height above the horizontal plane at which the collision occurs.

STEP III 1988 Question 14 (Mechanics)

14 A small heavy bead can slide smoothly in a vertical plane on a fixed wire with equation

$$y = x - \frac{x^2}{4a},$$

where the y-axis points vertically upwards and a is a positive constant. The bead is projected from the origin with initial speed V along the wire.

(i) Show that for a suitable value of V, to be determined, a motion is possible throughout which the bead exerts no pressure on the wire.

(ii) Show that θ, the angle between the particle's velocity at time t and the x-axis, satisfies

$$\frac{4a^2\dot{\theta}^2}{\cos^6\theta} + 2ga(1 - \tan^2\theta) = V^2.$$

STEP II 2002 Question 9 (Mechanics)

9 A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α. The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h, at distances a and b from O. Given that the particle just passes over the walls, find $\tan\alpha$ in terms of a, b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab} \, .$$

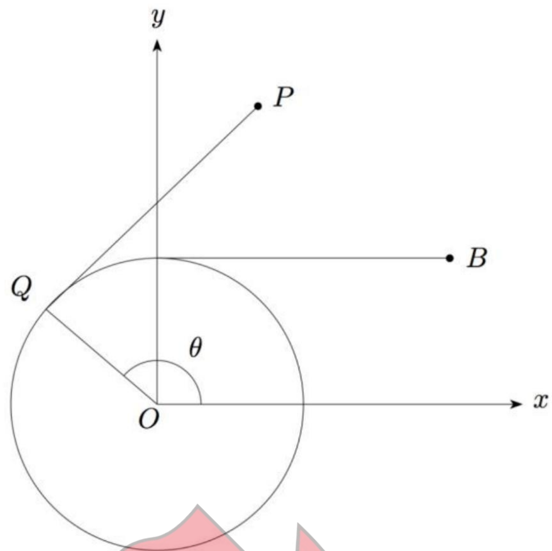
The heights of the walls are now increased by the same small positive amount δh. A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2\alpha \, \delta\alpha \approx \frac{a+b}{ab} \, \delta h \, ,$$

and deduce that $\delta\alpha > 0$. Show also that δV is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

STEP III 1992 Question 14 (Mechanics)

14



A horizontal circular disc of radius a and centre O lies on a horizontal table and is fixed to it so that it cannot rotate. A light inextensible string of negligible thickness is wrapped round the disc and attached at its free end to a particle P of mass m. When the string is all in contact with the disc, P is at A. The string is unwound so that the part not in contact with the disc is taut and parallel to OA. P is then at B. The particle is projected along the table from B with speed V perpendocular to and away from OA. In the general position, the string is tangential to the disc at Q and $\angle AOQ = \theta$. Show that, in the general position, the x-coordinate of P with respect to the axes shown in the figure is $a\cos\theta + a\theta\sin\theta$, and find y-coordinate of P. Hence, or otherwise, show that the acceleration of P has components $a\theta\dot{\theta}^2$ and $a\dot{\theta}^2 + a\theta\ddot{\theta}$ along and perpendicular to PQ, respectively.

The friction force between P and the table is $2\lambda mv^2/a$, where v is the speed of P and λ is a constant. Show that

$$\frac{\ddot{\theta}}{\dot{\theta}} = -\left(\frac{1}{\theta} + 2\lambda\theta\right)\dot{\theta}$$

and find $\dot{\theta}$ in terms of θ, λ and a. Find also the tension in the string when $\theta = \pi$.

STEP I 1994 Question 9 (Mechanics)

9 A cannon-ball is fired from a cannon at an initial speed u. After time t it has reached height h and is at a distance $\sqrt{x^2+h^2}$ from the cannon. Ignoring air resistance, show that

$$\tfrac{1}{4}g^2t^4 - (u^2 - gh)t^2 + h^2 + x^2 = 0.$$

Hence show that if $u^2 > 2gh$ then the horizontal range for a given height h and initial speed u is less than or equal to

$$\frac{u\sqrt{u^2-2gh}}{g}.$$

Show that there is always an angle of firing for which this value is attained.

STEP III 2014 Question 9 (Mechanics)

9 A particle of mass m is projected with velocity $\mathbf{u}$. It is acted upon by the force $m\mathbf{g}$ due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a positive constant.

Given that, at time t after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + \mathrm{e}^{-kt}}{k^2}\,\mathbf{g} + \frac{1 - \mathrm{e}^{-kt}}{k}\,\mathbf{u}\,,$$

find an expression for $\mathbf{v}$ in terms of k, t, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

Let $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\,\mathbf{j}$, where $0 < \alpha < 90°$, and let T be the time after projection at which $\mathbf{r}\,.\,\mathbf{j} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - \mathrm{e}^{-kT}} - 1\right) g\,.$$

Let β be the acute angle between $\mathbf{v}$ and $\mathbf{i}$ at time T. Show that

$$\tan\beta = \frac{(\mathrm{e}^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha\,.$$

Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.

STEP I 2001 Question 10 (Mechanics)

10 A gun is sited on a horizontal plain and can fire shells in any direction and at any elevation at speed v. The gun is a distance d from a straight railway line which crosses the plain, where $v^2 > gd$. The gunner aims to hit the line, choosing the direction and elevation so as to maximize the time of flight of the shell. Show that the time of flight, T, of the shell satisfies

$$g^2T^2 = 2v^2 + 2\left(v^4 - g^2d^2\right)^{\frac{1}{2}} .$$
