STEP Past Papers by Topic

STEP Topic – Proof by Induction

STEP II 1997 Question 2 (Pure)

2 Suppose that

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \cdots .$$

Guess an expression, in terms of n, for x_n. Then, by induction or otherwise, prove the correctness of your guess.

STEP III 1990 Question 5 (Pure)

5 Prove that, for any integers n and r, with $1 \leqslant r \leqslant n$,

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

Hence or otherwise, prove that

$$(uv)^{(n)} = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots + uv^{(n)},$$

where u and v are functions of x and $z^{(r)}$ means $\dfrac{\mathrm{d}^r z}{\mathrm{d}x^r}$.

Prove that, if $y = \sin^{-1} y$, then $(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - n^2y^{(n)} = 0$.

STEP III Specimen Question 1 (Pure)

1 **(i)** Guess an expression for

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\cdots\left(1-\frac{1}{n^2}\right),$$

valud for $n \geqslant 2$, and prove by mathematical induction the correctness of your guess.

(ii) Show that, if n is a positive integer,

$$\sum_{r=0}^{k}(-1)^r\binom{n}{r} = (-1)^k\binom{n-1}{k}, \qquad \text{for } k = 0, 1, \ldots, n-1.$$

STEP II 1999 Question 3 (Pure)

3 Let

$$\mathrm{S}_n(x) = \mathrm{e}^{x^3} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^3}).$$

Show that $\mathrm{S}_2(x) = 9x^4 - 6x$ and find $\mathrm{S}_3(x)$.

Prove by induction on n that $\mathrm{S}_n(x)$ is a polynomial. By means of your induction argument, determine the order of this polynomial and the coefficient of the highest power of x.

Show also that if $\dfrac{\mathrm{d}S_n}{\mathrm{d}x} = 0$ for some value a of x, then $S_n(a)S_{n+1}(a) \leqslant 0$.

STEP III 2000 Question 8 (Pure)

8 The sequence a_n is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \qquad (n \geqslant 2).$$

Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \qquad (n \geqslant 2).$$

Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \qquad (n \geqslant 1),$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

STEP I 1994 Question 7 (Pure)

7 From the facts

$$\begin{aligned} 1 &= 0 \\ 2+3+4 &= 1+8 \\ 5+6+7+8+9 &= 8+27 \\ 10+11+12+13+14+15+16 &= 27+64 \end{aligned}$$

guess a general law. Prove it.

Hence, or otherwise, prove that

$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \tfrac{1}{4}N^2(N+1)^2$$

for every positive integer N.

[**Hint.** *You may assume that* $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$.]

STEP I 1995 Question 3 (Pure)

3 **(i)** If $\mathrm{f}(r)$ is a function defined for $r = 0, 1, 2, 3, \ldots$, show that

$$\sum_{r=1}^{n} \{\mathrm{f}(r) - \mathrm{f}(r-1)\} = \mathrm{f}(n) - \mathrm{f}(0).$$

(ii) If $\mathrm{f}(r) = r^2(r+1)^2$, evaluate $\mathrm{f}(r) - \mathrm{f}(r-1)$ and hence determine $\displaystyle\sum_{r=1}^{n} r^3$.

(iii) Find the sum of the series $1^3 - 2^3 + 3^3 - 4^3 + \cdots + (2n+1)^3$.

STEP II 1994 Question 6 (Pure)

6 Prove by induction, or otherwise, that, if $0 < \theta < \pi$,

$$\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \cot\theta.$$

Deduce that

$$\sum_{r=1}^{\infty}\frac{1}{2^r}\tan\frac{\theta}{2^r} = \frac{1}{\theta} - \cot\theta.$$

STEP III 2009 Question 7 (Pure)

7 **(i)** The functions $\mathrm{f}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{f}_0(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad \mathrm{f}_{n+1}(x) = \frac{\mathrm{d}\mathrm{f}_n(x)}{\mathrm{d}x}\,.$$

Prove, for $n \geqslant 1$, that

$$(1+x^2)\mathrm{f}_{n+1}(x) + 2(n+1)x\mathrm{f}_n(x) + n(n+1)\mathrm{f}_{n-1}(x) = 0\,.$$

(ii) The functions $\mathrm{P}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{P}_n(x) = (1+x^2)^{n+1}\mathrm{f}_n(x)\,.$$

Find expressions for $\mathrm{P}_0(x)$, $\mathrm{P}_1(x)$ and $\mathrm{P}_2(x)$.

Prove, for $n \geqslant 0$, that

$$\mathrm{P}_{n+1}(x) - (1+x^2)\frac{\mathrm{d}\mathrm{P}_n(x)}{\mathrm{d}x} + 2(n+1)x\mathrm{P}_n(x) = 0\,,$$

and that $\mathrm{P}_n(x)$ is a polynomial of degree n.

STEP III 2011 Question 7 (Pure)

7 Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a}\right)^n ,$$

where n is a positive integer and a is any given positive integer.

(i) In the case when n is even, show by induction that T_n can be written in the form

$$A_n + B_n\sqrt{a(a+1)} ,$$

where A_n and B_n are integers (depending on a and n) and $A_n^2 = a(a+1)B_n^2 + 1$.

(ii) In the case when n is odd, show by considering $(\sqrt{a+1} + \sqrt{a})T_m$ where m is even, or otherwise, that T_n can be written in the form

$$C_n\sqrt{a+1} + D_n\sqrt{a} ,$$

where C_n and D_n are integers (depending on a and n) and $(a+1)C_n^2 = aD_n^2 + 1$.

(iii) Deduce that, for each n, T_n can be written as the sum of the square roots of two consecutive integers.

STEP II 2015 Question 3 (Pure)

3 Three rods have lengths a, b and c, where $a < b < c$. The three rods can be made into a triangle (possibly of zero area) if $a + b \geqslant c$.

Let T_n be the number of triangles that can be made with three rods chosen from n rods of lengths $1, 2, 3, \ldots, n$ (where $n \geqslant 3$). Show that $T_8 - T_7 = 2 + 4 + 6$ and evaluate $T_8 - T_6$. Write down expressions for $T_{2m} - T_{2m-1}$ and $T_{2m} - T_{2m-2}$.

Prove by induction that $T_{2m} = \frac{1}{6}m(m-1)(4m+1)$, and find the corresponding result for an odd number of rods.
