STEP Past Papers by Topic

STEP Topic – Trigonometry

STEP III Specimen Question 5 (Pure)

5 The equation

$$\sin x = \lambda x, \qquad x \geqslant 0,$$

where $\lambda > 0$, has a finite number N of non-zero solutions x_n, $i = 1, \ldots, N$, where N depends on λ, provided $\lambda < 1$.

(i) Show by a graphical argument that there are no non-zero solutions for $\lambda > 1$. Show also that for $\lambda = 1 - \epsilon^2$, with $\epsilon > 0$ and very small compared to 1, there is a non-zero solution approximately equal to $\epsilon\sqrt{6}$.

(ii) Suppose that $N = 2R + 1$ where R is an integer, and that $x_1 < x_2 < \cdots < x_{2R+1}$. By drawing an appropriate graph, explain why

$$(2n-2)\pi < x_{2n-1} < (2n-1)\pi \quad \text{for } n = 1, \ldots, R+1,$$
$$2n\pi < x_{2n} < (2n+\tfrac{1}{2})\pi \quad \text{for } n = 1, \ldots, R.$$

Hence derive an approximate value for N in terms of λ, when λ is very small.

STEP II 1988 Question 5 (Pure)

5 By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

STEP III 1990 Question 4 (Pure)

4 Given that $\sin\beta \neq 0$, sum the series

$$\cos\alpha + \cos(\alpha + 2\beta) + \cdots + \cos(\alpha + 2r\beta) + \cdots + \cos(\alpha + 2n\beta)$$

and

$$\cos\alpha + \binom{n}{1}\cos(\alpha + 2\beta) + \cdots + \binom{n}{r}\cos(\alpha + 2r\beta) + \cdots + \cos(\alpha + 2n\beta).$$

Given that $\sin\theta \neq 0$, prove that

$$1 + \cos\theta\sec\theta + \cos 2\theta\sec^2\theta + \cdots + \cos r\theta\sec^r\theta + \cdots + \cos n\theta\sec^n\theta = \frac{\sin(n+1)\theta\sec^n\theta}{\sin\theta}.$$

STEP II 1991 Question 4 (Pure)

4 Let $y = \cos\phi + \cos 2\phi$, where $\phi = \dfrac{2\pi}{5}$. Verify by direct substitution that y satisfies the quadratic equation $2y^2 = 3y + 2$ and deduce that the value of y is $-\frac{1}{2}$.

Let $\theta = \dfrac{2\pi}{17}$. Show that

$$\sum_{k=0}^{16} \cos k\theta = 0.$$

If $z = \cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, show that the value of z is $-(1 - \sqrt{17})/4$.

STEP III 1988 Question 6 (Pure)

6 Let $\mathrm{f}(x) = \sin 2x \cos x$. Find the 1988th derivative of $\mathrm{f}(x)$.

Show that the smallest positive value of x for which this derivative is zero is $\frac{1}{3}\pi + \epsilon$, where ϵ is approximately equal to

$$\frac{3^{-1988}\sqrt{3}}{2}.$$

STEP II 1987 Question 2 (Pure)

2 Show that if at least one of the four angles $A \pm B \pm C$ is a multiple of π, then

$$\sin^4 A + \sin^4 B + \sin^4 C - 2\sin^2 B \sin^2 C - 2\sin^2 C \sin^2 A \\ - 2\sin^2 A \sin^2 B + 4\sin^2 A \sin^2 B \sin^2 C = 0.$$

STEP II 1990 Question 2 (Pure)

2 Prove that if $A + B + C + D = \pi$, then

$$\sin(A+B)\sin(A+D) - \sin B \sin D = \sin A \sin C.$$

The points P, Q, R and S lie, in that order, on a circle of centre O. Prove that

$$PQ \times RS + QR \times PS = PR \times QS.$$

STEP I 1987 Question 7 (Pure)

7 Sum each of the series

$$\sin\left(\frac{2\pi}{23}\right)+\sin\left(\frac{6\pi}{23}\right)+\sin\left(\frac{10\pi}{23}\right)+\cdots+\sin\left(\frac{38\pi}{23}\right)+\sin\left(\frac{42\pi}{23}\right)$$

and

$$\sin\left(\frac{2\pi}{23}\right)-\sin\left(\frac{6\pi}{23}\right)+\sin\left(\frac{10\pi}{23}\right)-\cdots-\sin\left(\frac{38\pi}{23}\right)+\sin\left(\frac{42\pi}{23}\right),$$

giving each answer in terms of the tangent of a single angle.

[No credit will be given for a numerical answer obtained purely by use of a calculator.]

STEP I 1991 Question 1 (Pure)

1 If $\theta + \phi + \psi = \frac{1}{2}\pi$, show that

$$\sin^2\theta + \sin^2\phi + \sin^2\psi + 2\sin\theta\sin\phi\sin\psi = 1.$$

By taking $\theta = \phi = \frac{1}{5}\pi$ in this equation, or otherwise, show that $\sin\frac{1}{10}\pi$ satisfies the equation

$$8x^3 + 8x^2 - 1 = 0.$$

STEP II 2002 Question 4 (Pure)

4 Give a sketch to show that, if $f(x) > 0$ for $p < x < q$, then $\int_p^q f(x)dx > 0$.

(i) By considering $f(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $f(x) = a\sin^2 x - b\sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b\ln(q/p) < a\left(\frac{1}{p} - \frac{1}{q}\right) + c(q - p) .$$

STEP III 2004 Question 5 (Pure)

5 Show that if $\cos(x-\alpha) = \cos\beta$ then either $\tan x = \tan(\alpha+\beta)$ or $\tan x = \tan(\alpha-\beta)$. By choosing suitable values of x, α and β, give an example to show that if $\tan x = \tan(\alpha+\beta)$, then $\cos(x-\alpha)$ need not equal $\cos\beta$.

Let ω be the acute angle such that $\tan\omega = \frac{4}{3}$.

(i) For $0 \leqslant x \leqslant 2\pi$, solve the equation

$$\cos x - 7\sin x = 5$$

giving both solutions in terms of ω.

(ii) For $0 \leqslant x \leqslant 2\pi$, solve the equation

$$2\cos x + 11\sin x = 10$$

showing that one solution is twice the other and giving both in terms of ω.

STEP II 1997 Question 6 (Pure)

6 Show that, if $\tan^2\phi = 2\tan\phi + 1$, then $\tan 2\phi = -1$.

Find all solutions of the equation

$$\tan\theta = 2 + \tan 3\theta$$

which satisfy $0 < \theta < 2\pi$, expressing your answers as rational multiples of π.

Find all solutions of the equation the equation

$$\cot\theta = 2 + \cot 3\theta$$

which satisfy

$$-\frac{3\pi}{2} < \theta < \frac{\pi}{2}.$$

STEP I 2005 Question 4 (Pure)

4 **(i)** Given that $\cos\theta = \dfrac{3}{5}$ and that $\dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\dfrac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \dfrac{11}{2}$ and that $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

STEP II 1999 Question 5 (Pure)

5 Show that if α is a solution of the equation

$$5\cos x + 12\sin x = 7,$$

then either

$$\cos\alpha = \frac{35 - 12\sqrt{120}}{169}$$

or $\cos\alpha$ has one other value which you should find.

Prove carefully that if $\frac{1}{2}\pi < \alpha < \pi$, then $\alpha < \frac{3}{4}\pi$.

STEP II 2005 Question 4 (Pure)

4 The positive numbers a, b and c satisfy $bc = a^2 + 1$. Prove that

$$\tan^{-1}\left(\frac{1}{a+b}\right) + \tan^{-1}\left(\frac{1}{a+c}\right) = \tan^{-1}\left(\frac{1}{a}\right).$$

The positive numbers p, q, r, s, t, u and v satisfy

$$st = (p+q)^2 + 1\,, \qquad uv = (p+r)^2 + 1\,, \qquad qr = p^2 + 1\,.$$

Prove that

$$\tan^{-1}\left(\frac{1}{p+q+s}\right) + \tan^{-1}\left(\frac{1}{p+q+t}\right) + \tan^{-1}\left(\frac{1}{p+r+u}\right) + \tan^{-1}\left(\frac{1}{p+r+v}\right) = \tan^{-1}\left(\frac{1}{p}\right).$$

Hence show that

$$\tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \tan^{-1}\left(\frac{1}{82}\right) + \tan^{-1}\left(\frac{1}{187}\right) = \tan^{-1}\left(\frac{1}{7}\right).$$

[Note that $\arctan x$ is another notation for $\tan^{-1} x$.]

STEP III 2007 Question 1 (Pure)

1 *In this question, do not consider the special cases in which the denominators of any of your expressions are zero.*

Express $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$ in terms of t_i, where $t_1 = \tan\theta_1$, etc.

Given that $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the four roots of the equation

$$at^4 + bt^3 + ct^2 + dt + e = 0$$

(where $a \neq 0$), find an expression in terms of a, b, c, d and e for $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$.

The four real numbers θ_1, θ_2, θ_3 and θ_4 lie in the range $0 \leqslant \theta_i < 2\pi$ and satisfy the equation

$$p\cos 2\theta + \cos(\theta - \alpha) + p = 0\,,$$

where p and α are independent of θ. Show that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = n\pi$ for some integer n.

STEP I 2010 Question 3 (Pure)

3 Show that

$$\sin(x+y) - \sin(x-y) = 2\cos x \, \sin y$$

and deduce that

$$\sin A - \sin B = 2\cos \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

Show also that

$$\cos A - \cos B = -2\sin \tfrac{1}{2}(A+B) \, \sin \tfrac{1}{2}(A-B)\,.$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

STEP I 2003 Question 3 (Pure)

3 **(i)** Show that $2\sin(\frac{1}{2}\theta) = \sin\theta$ if and only if $\sin(\frac{1}{2}\theta) = 0$.

(ii) Solve the equation $2\tan(\frac{1}{2}\theta) = \tan\theta$.

(iii) Show that $2\cos(\frac{1}{2}\theta) = \cos\theta$ if and only if $\theta = (4n+2)\pi \pm 2\phi$ where ϕ is defined by $\cos\phi = \frac{1}{2}(\sqrt{3}-1)$, $0 \leqslant \phi \leqslant \frac{1}{2}\pi$, and n is any integer.

STEP II 2007 Question 4 (Pure)

4 Given that $\cos A$, $\cos B$ and β are non-zero, show that the equation

$$\alpha \sin(A - B) + \beta \cos(A + B) = \gamma \sin(A + B)$$

reduces to the form

$$(\tan A - m)(\tan B - n) = 0\,,$$

where m and n are independent of A and B, if and only if $\alpha^2 = \beta^2 + \gamma^2$.

Determine all values of x, in the range $0 \leqslant x < 2\pi$, for which:

(i) $2\sin(x - \frac{1}{4}\pi) + \sqrt{3}\cos(x + \frac{1}{4}\pi) = \sin(x + \frac{1}{4}\pi)\,;$

(ii) $2\sin(x - \frac{1}{6}\pi) + \sqrt{3}\cos(x + \frac{1}{6}\pi) = \sin(x + \frac{1}{6}\pi)\,;$

(iii) $2\sin(x + \frac{1}{3}\pi) + \sqrt{3}\cos(3x) = \sin(3x)\,.$

STEP II 2001 Question 4 (Pure)

4 Let

$$f(x) = P \sin x + Q \sin 2x + R \sin 3x \, .$$

Show that if $Q^2 < 4R(P - R)$, then the only values of x for which $f(x) = 0$ are given by $x = m\pi$, where m is an integer.
[You may assume that $\sin 3x = \sin x(4\cos^2 x - 1)$.]

Now let

$$g(x) = \sin 2nx + \sin 4nx - \sin 6nx,$$

where n is a positive integer and $0 < x < \frac{1}{2}\pi$. Find an expression for the largest root of the equation $g(x) = 0$, distinguishing between the cases where n is even and n is odd.

STEP II 2003 Question 2 (Pure)

2 Write down a value of θ in the interval $\frac{1}{4}\pi < \theta < \frac{1}{2}\pi$ that satisfies the equation

$$4\cos\theta + 2\sqrt{3}\,\sin\theta = 5\,.$$

Hence, or otherwise, show that

$$\pi = 3\arccos(5/\sqrt{28}) + 3\arctan(\sqrt{3}/2)\,.$$

Show that

$$\pi = 4\arcsin(7\sqrt{2}/10) - 4\arctan(3/4)\,.$$

STEP I 2007 Question 2 (Pure)

2 **(i)** Given that $A = \arctan \frac{1}{2}$ and that $B = \arctan \frac{1}{3}$ (where A and B are acute) show, by considering $\tan(A + B)$, that $A + B = \frac{1}{4}\pi$.

The non-zero integers p and q satisfy

$$\arctan \frac{1}{p} + \arctan \frac{1}{q} = \frac{\pi}{4}\,.$$

Show that $(p-1)(q-1) = 2$ and hence determine p and q.

(ii) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan \frac{1}{r} + \arctan \frac{s}{s+t} = \frac{\pi}{4}\,,$$

then there are only two possible values for t, and give r in terms of s in each case.

STEP I 2001 Question 4 (Pure)

4 Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Given that $\theta = \cos^{-1}(2/\sqrt{5})$ and $0 < \theta < \pi/2$, show that $\tan 3\theta = 11/2$.

Hence, or otherwise, find all solutions of the equations

(i) $\tan(3\cos^{-1} x) = 11/2$,

(ii) $\cos(\frac{1}{3}\tan^{-1} y) = 2/\sqrt{5}$.

STEP III 2002 Question 2 (Pure)

2 Prove that $\arctan a + \arctan b = \arctan\left(\dfrac{a+b}{1-ab}\right)$ when $0 < a < 1$ and $0 < b < 1$.

Prove by induction that, for $n \geqslant 1$,

$$\sum_{r=1}^{n} \arctan\left(\frac{1}{r^2+r+1}\right) = \arctan\left(\frac{n}{n+2}\right)$$

and hence find

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2+r+1}\right).$$

Hence prove that

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{r^2-r+1}\right) = \frac{\pi}{2}.$$

STEP I 2015 Question 2 (Pure)

2 **(i)** Show that $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and find a similar expression for $\sin 15°$.

(ii) Show that $\cos\alpha$ is a root of the equation

$$4x^3 - 3x - \cos 3\alpha = 0\,,$$

and find the other two roots in terms of $\cos\alpha$ and $\sin\alpha$.

(iii) Use parts (i) and (ii) to solve the equation $y^3 - 3y - \sqrt{2} = 0$, giving your answers in surd form.

STEP II 2011 Question 4 (Pure)

4 **(i)** Find all the values of θ, in the range $0^\circ < \theta < 180^\circ$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18^\circ = \frac{1}{4}\left(\sqrt{5} - 1\right).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x\,,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0^\circ < \alpha < 90^\circ$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha\,.$$

STEP II 2009 Question 3 (Pure)

3 Prove that

$$\tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}x\right) \equiv \sec x - \tan x\,. \qquad (*)$$

(i) Use $(*)$ to find the value of $\tan\frac{1}{8}\pi$. Hence show that

$$\tan\tfrac{11}{24}\pi = \frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1}\,.$$

(ii) Show that

$$\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1} = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}\,.$$

(iii) Use $(*)$ to show that

$$\tan\tfrac{1}{48}\pi = \sqrt{16+10\sqrt{2}+8\sqrt{3}+6\sqrt{6}} - 2 - \sqrt{2} - \sqrt{3} - \sqrt{6}\,.$$

STEP I 2011 Question 3 (Pure)

3 Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

(i) By differentiating $(*)$, or otherwise, show that

$$\cot\tfrac{1}{9}\pi-\cot\tfrac{2}{9}\pi+\cot\tfrac{4}{9}\pi=\sqrt{3}\,.$$

(ii) By setting $\theta=\frac{1}{6}\pi-\phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi-\theta)\cot(\tfrac{1}{3}\pi+\theta)=\cot 3\theta\,.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi-\operatorname{cosec}\tfrac{5}{9}\pi+\operatorname{cosec}\tfrac{7}{9}\pi=2\sqrt{3}\,.$$

STEP II 2008 Question 6 (Pure)

6 A curve has the equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of $\mathrm{f}(x)$.

(ii) Determine all values of x in the interval $-\pi \leqslant x \leqslant \pi$ for which $\mathrm{f}(x) = 0$. Find a value of x in this interval at which the curve touches the x-axis without crossing it.

(iii) Find the value or values of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $\mathrm{f}(x) = 2$.
