STEP Past Papers by Topic

STEP Topic – Vector Geometry

STEP I 2016 Question 6 (Pure Mathematics)

6 The sides OA and CB of the quadrilateral $OABC$ are parallel. The point X lies on OA, between O and A. The position vectors of A, B, C and X relative to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively. Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \quad \text{and} \quad \mathbf{x} = m\mathbf{a}\,,$$

where k and m are scalars, and state the range of values that each of k and m can take.

The lines OB and AC intersect at D, the lines XD and BC intersect at Y and the lines OY and AB intersect at Z. Show that the position vector of Z relative to O can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk+1}\,.$$

The lines DZ and OA intersect at T. Show that

$$OT \times OA = OX \times TA \quad \text{and} \quad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA}\,,$$

where, for example, OT denotes the length of the line joining O and T.

STEP II 2017 Question 8 (Pure)

All vectors in this question lie in the same plane.

The vertices of the non-right-angled triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to BC and CA, respectively.

Write down the vector equation of the line through A perpendicular to BC, in terms of $\mathbf{u}$, $\mathbf{a}$ and a parameter λ.

The line through A perpendicular to BC intersects the line through B perpendicular to CA at P. Find the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{u}$.

Hence show that the line CP is perpendicular to the line AB.

STEP II 2003 Question 5 (Pure)

5 The position vectors of the points A, B and P with respect to an origin O are $a\mathbf{i}$, $b\mathbf{j}$ and $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, respectively, where a, b, and n are all non-zero. The points E, F, G and H are the midpoints of OA, BP, OB and AP, respectively. Show that the lines EF and GH intersect.

Let D be the point with position vector $d\mathbf{k}$, where d is non-zero, and let S be the point of intersection of EF and GH. The point T is such that the mid-point of DT is S. Find the position vector of T and hence find d in terms of n if T lies in the plane OAB.

STEP I 2014 Question 7 (Pure)

7 In the triangle OAB, the point D divides the side BO in the ratio $r:1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s:1$ (so that $OE = sEA$), where r and s are both positive.

(i) The lines AD and BE intersect at G. Show that

$$\mathbf{g} = \frac{rs}{1+r+rs}\,\mathbf{a} + \frac{1}{1+r+rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t:1$, find an expression for t in terms of r and s.

STEP II 2008 Question 8 (Pure)

8 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1 - \lambda) : \lambda \,.$$

Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b - a)^2 \,.$$

STEP II 2011 Question 5 (Pure)

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \dfrac{2\,\mathbf{a.b}}{\mathbf{a.a}}$.

The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

STEP III 2013 Question 3 (Pure)

3 The four vertices P_i ($i = 1, 2, 3, 4$) of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ ($i = 1, 2, 3, 4$). Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i ($i = 1, 2, 3, 4$).

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8 \,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2 \,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum\limits_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

STEP II 2007 Question 8 (Pure)

8 The points B and C have position vectors $\mathbf{b}$ and $\mathbf{c}$, respectively, relative to the origin A, and A, B and C are not collinear.

(i) The point X has position vector $s\mathbf{b} + t\mathbf{c}$. Describe the locus of X when $s + t = 1$.

(ii) The point P has position vector $\beta\mathbf{b} + \gamma\mathbf{c}$, where β and γ are non-zero, and $\beta + \gamma \neq 1$. The line AP cuts the line BC at D. Show that $BD : DC = \gamma : \beta$.

(iii) The line BP cuts the line CA at E, and the line CP cuts the line AB at F. Show that

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1\,.$$

STEP I 2010 Question 7 (Pure)

7 Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of $\alpha,\ \beta$ and $\mathbf{b}$.

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta}\,,$$

lies on the lines OC and AB.

The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

STEP II 2009 Question 8 (Pure)

8 The non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, given by

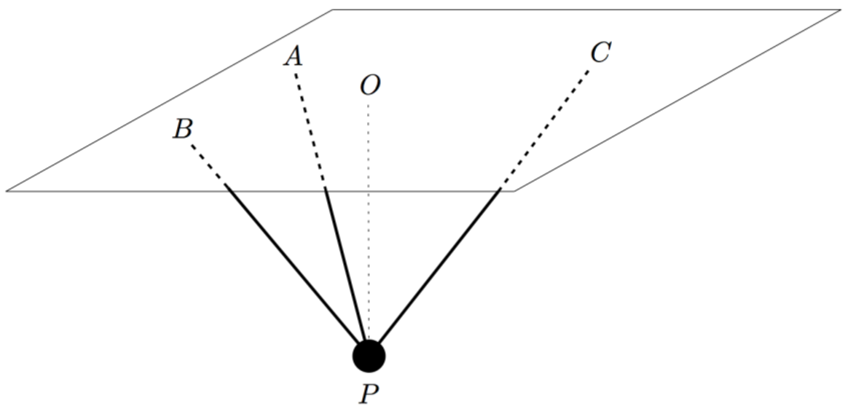
$$\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \quad \text{and} \quad \mathbf{q} = \mu\mathbf{a} + (1-\mu)\mathbf{c}$$

where $0 < \lambda < 1$ and $\mu > 1$. Draw a diagram showing A, B, C, P and Q.

Given that $CQ \times BP = AB \times AC$, find μ in terms of λ, and show that, for all values of λ, the the line PQ passes through the fixed point D, with position vector $\mathbf{d}$ given by $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$. What can be said about the quadrilateral $ABDC$?

STEP II 2011 Question 11 (Mechanics)

11 Three non-collinear points A, B and C lie in a horizontal ceiling. A particle P of weight W is suspended from this ceiling by means of three light inextensible strings AP, BP and CP, as shown in the diagram. The point O lies vertically above P in the ceiling.



The angles AOB and AOC are $90° + \theta$ and $90° + \phi$, respectively, where θ and ϕ are acute angles such that $\tan\theta = \sqrt{2}$ and $\tan\phi = \frac{1}{4}\sqrt{2}$.

The strings AP, BP and CP make angles $30°$, $90° - \theta$ and $60°$, respectively, with the vertical, and the tensions in these strings have magnitudes T, U and V respectively.

(i) Show that the unit vector in the direction PB can be written in the form

$$-\frac{1}{3}\mathbf{i} - \frac{\sqrt{2}}{3}\mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}}\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual mutually perpendicular unit vectors with $\mathbf{j}$ parallel to OA and $\mathbf{k}$ vertically upwards.

(ii) Find expressions in vector form for the forces acting on P.

(iii) Show that $U = \sqrt{6}V$ and find T, U and V in terms of W.

STEP II 2006 Question 8 (Pure)

8 Show that the line through the points with position vectors $\mathbf{x}$ and $\mathbf{y}$ has equation

$$\mathbf{r} = (1-\alpha)\mathbf{x} + \alpha\mathbf{y}\,,$$

where α is a scalar parameter.

The sides OA and CB of a trapezium $OABC$ are parallel, and $OA > CB$. The point E on OA is such that $OE : EA = 1 : 2$, and F is the midpoint of CB. The point D is the intersection of OC produced and AB produced; the point G is the intersection of OB and EF; and the point H is the intersection of DG produced and OA. Let $\mathbf{a}$ and $\mathbf{c}$ be the position vectors of the points A and C, respectively, with respect to the origin O.

(i) Show that B has position vector $\lambda\mathbf{a} + \mathbf{c}$ for some scalar parameter λ.

(ii) Find, in terms of $\mathbf{a}$, $\mathbf{c}$ and λ only, the position vectors of D, E, F, G and H. Determine the ratio $OH : HA$.

STEP II 2012 Question 7 (Pure)

7 Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly.

Given that $\overrightarrow{GY_1} = -\lambda_1\overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1} = \tfrac{1}{3}\big((1-2\lambda_1)\mathbf{x}_1 + (1+\lambda_1)(\mathbf{x}_2+\mathbf{x}_3)\big)$$

and hence that

$$\lambda_1 = \frac{3-\alpha-\beta-\gamma}{3+\alpha-2\beta-2\gamma}\,,$$

where $\alpha = \mathbf{x}_2\,.\,\mathbf{x}_3$, $\beta = \mathbf{x}_3\,.\,\mathbf{x}_1$ and $\gamma = \mathbf{x}_1\,.\,\mathbf{x}_2$.

Deduce that $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = 3\,.$

STEP II 2010 Question 5 (Pure)

5 The points A and B have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $5\mathbf{i}-\mathbf{j}-\mathbf{k}$, respectively, relative to the origin O. Find $\cos 2\alpha$, where 2α is the angle $\angle AOB$.

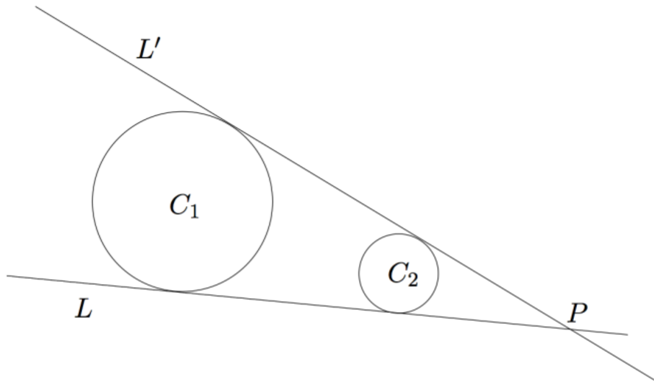
(i) The line L_1 has equation $\mathbf{r} = \lambda(m\mathbf{i}+n\mathbf{j}+p\mathbf{k})$. Given that L_1 is inclined equally to OA and to OB, determine a relationship between m, n and p. Find also values of m, n and p for which L_1 is the angle bisector of $\angle AOB$.

(ii) The line L_2 has equation $\mathbf{r} = \mu(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})$. Given that L_2 is inclined at an angle α to OA, where $2\alpha = \angle AOB$, determine a relationship between u, v and w.

Hence describe the surface with Cartesian equation $x^2+y^2+z^2 = 2(yz+zx+xy)$.

STEP II 2015 Question 8 (Pure)

8



The diagram above shows two non-overlapping circles C_1 and C_2 of different sizes. The lines L and L' are the two common tangents to C_1 and C_2 such that the two circles lie on the same side of each of the tangents. The lines L and L' intersect at the point P which is called the *focus* of C_1 and C_2.

(i) Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the position vectors of the centres of C_1 and C_2, respectively. Show that the position vector of P is

$$\frac{r_1\mathbf{x}_2 - r_2\mathbf{x}_1}{r_1 - r_2},$$

where r_1 and r_2 are the radii of C_1 and C_2, respectively.

(ii) The circle C_3 does not overlap either C_1 or C_2 and its radius, r_3, satisfies $r_1 \neq r_3 \neq r_2$. The focus of C_1 and C_3 is Q, and the focus of C_2 and C_3 is R. Show that P, Q and R lie on the same straight line.

(iii) Find a condition on r_1, r_2 and r_3 for Q to lie half-way between P and R.

STEP II 1988 Question 10 (Pure)

10 The surface S in 3-dimensional space is described by the equation

$$\mathbf{a}\cdot\mathbf{r} + ar = a^2,$$

where $\mathbf{r}$ is the position vector with respect to the origin O, $\mathbf{a}(\neq \mathbf{0})$ is the position vector of a fixed point, $r = |\mathbf{r}|$ and $a = |\mathbf{a}|$. Show, with the aid of a diagram, that S is the locus of points which are equidistant from the origin O and the plane $\mathbf{r}\cdot\mathbf{a} = a^2$.

The point P, with position vector $\mathbf{p}$, lies in S, and the line joining P to O meets S again at Q. Find the position vector of Q.

The line through O orthogonal to $\mathbf{p}$ and $\mathbf{a}$ meets S at T and T'. Show that the position vectors of T and T' are

$$\pm\frac{1}{\sqrt{2ap - a^2}}\mathbf{a}\times\mathbf{p},$$

where $p = |\mathbf{p}|$.

Show that the area of the triangle PQT is

$$\frac{ap^2}{2p - a}.$$

STEP I 1989 Question 3 (Pure)

3 In the triangle OAB, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $OA = OB = 1$. Points C and D trisect AB (i.e. $AC = CD = DB = \frac{1}{3}AB$). X and Y lie on the line-segments OA and OB respectively, in such a way that CY and DX are perpendicular, and $OX + OY = 1$. Denoting OX by x, obtain a condition relating x and $\mathbf{a} \cdot \mathbf{b}$, and prove that

$$\frac{8}{17} \leqslant \mathbf{a} \cdot \mathbf{b} \leqslant 1.$$

If the angle AOB is as large as possible, determine the distance OE, where E is the point of intersection of CY and DX.

STEP II 1989 Question 10 (Pure)

10 State carefully the conditions which the fixed vectors $\mathbf{a}, \mathbf{b}, \mathbf{u}$ and $\mathbf{v}$ must satisfy in order to ensure that the line $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$ intersects the line $\mathbf{r} = \mathbf{b} + \mu\mathbf{v}$ in exactly one point.

Find the two values of the fixed scalar b for which the planes with equations

$$\left.\begin{aligned} x + y + bz &= b + 2 \\ bx + by + z &= 2b + 1 \end{aligned}\right\} \qquad (*)$$

do not intersect in a line. For other values of b, express the line of intersection of the two planes in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$, where $\mathbf{a} \cdot \mathbf{u} = 0$.

Find the conditions which b and the fixed scalars c and d must satisfy to ensure that there is exactly one point on the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} + \mu \begin{pmatrix} 1 \\ d \\ 0 \end{pmatrix}$$

whose coordinates satisfy both equations $(*)$.

STEP II Specimen Question 9 (Pure)

9 **(i)** Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(ii) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

STEP I 1990 Question 6 (Pure)

6 Let $ABCD$ be a parallelogram. By using vectors, or otherwise, prove that:

(i) $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$;

(ii) $\left|AC^2 - BD^2\right|$ is 4 times the area of the rectangle whose sides are *any* side of the parallelogram and the projection of an adjacent side on that side.

State and prove a result like **(ii)** about $\left|AB^2 - AD^2\right|$ and the diagonals.

STEP II 1993 Question 4 (Pure)

4 Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2, 1, -1) \qquad \mathbf{p}_2 = (1, 0, -2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4, 3, 0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0, -3, 1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2 \, |v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

STEP III 1995 Question 8 (Pure)

8 A plane π in 3-dimensional space is given by the vector equation $\mathbf{r} \cdot \mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r} \cdot \mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda \mathbf{n}_1 = \mathbf{r}_2 \pm \lambda \mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1 \cdot \mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2 \cdot \mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.

STEP III 1998 Question 8 (Pure)

8 **(i)** Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r} \cdot \mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}.\mathbf{b} - (\mathbf{b} \cdot \mathbf{m})^2 \,.$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m} \cdot \mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d} \cdot \mathbf{d} = 2a^2 \,.$$

STEP II 1998 Question 8 (Pure)

8 Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in three dimensions have coordinate vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Show that the lines joining the vertices of the triangle ABC to the mid-points of the opposite sides meet at a point R.

P is a point which is **not** in the plane ABC. Lines are drawn through the mid-points of BC, CA and AB parallel to PA, PB and PC respectively. Write down the vector equations of the lines and show by inspection that these lines meet at a common point Q.

Prove further that the line PQ meets the plane ABC at R.

STEP II 2000 Question 7 (Pure)

7 The line l has vector equation $\mathbf{r} = \lambda\mathbf{s}$, where

$$\mathbf{s} = (\cos\theta + \sqrt{3})\,\mathbf{i} + (\sqrt{2}\ \sin\theta)\,\mathbf{j} + (\cos\theta - \sqrt{3})\,\mathbf{k}$$

and λ is a scalar parameter. Find an expression for the angle between l and the line $\mathbf{r} = \mu(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k})$. Show that there is a line m through the origin such that, whatever the value of θ, the acute angle between l and m is $\pi/6$.

A plane has equation $x - z = 4\sqrt{3}$. The line l meets this plane at P. Show that, as θ varies, P describes a circle, with its centre on m. Find the radius of this circle.

STEP II 2004 Question 6 (Pure)

6 The vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the plane Π. Given that $|\mathbf{a}| = 1$ and $\mathbf{a.b} = 3$, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, a vector $\mathbf{p}$ parallel to $\mathbf{a}$ and a vector $\mathbf{q}$ perpendicular to $\mathbf{a}$, both lying in the plane Π, such that

$$\mathbf{p} + \mathbf{q} = \mathbf{a} + \mathbf{b}\,.$$

The vector $\mathbf{c}$ is not parallel to the plane Π and is such that $\mathbf{a.c} = -2$ and $\mathbf{b.c} = 2$. Given that $|\mathbf{b}| = 5$, find, in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ such that $\mathbf{P}$ and $\mathbf{Q}$ are parallel to $\mathbf{p}$ and $\mathbf{q}$, respectively, $\mathbf{R}$ is perpendicular to the plane Π and

$$\mathbf{P} + \mathbf{Q} + \mathbf{R} = \mathbf{a} + \mathbf{b} + \mathbf{c}\,.$$

STEP I 1995 Question 7 (Pure)

7 Let A, B, C be three non-collinear points in the plane. Explain briefly why it is possible to choose an origin equidistant from the three points. Let O be such an origin, let G be the centroid of the triangle ABC, let Q be a point such that $\overrightarrow{GQ} = 2\overrightarrow{OG}$, and let N be the midpoint of OQ.

(i) Show that $\overrightarrow{AQ}$ is perpendicular to $\overrightarrow{BC}$ and deduce that the three altitudes of $\triangle ABC$ are concurrent.

(ii) Show that the midpoints of AQ, BQ and CQ, and the midpoints of the sides of $\triangle ABC$ are all equidistant from N.

[The *centroid* of $\triangle ABC$ is the point G such that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$. The *altitudes* of the triangle are the lines through the vertices perpendicular to the opposite sides.]
