FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics 11

Key messages

- Candidates should read each question carefully so that they use all the information given and answer all aspects in adequate depth. They should make clear the method being used.
- All sketch graphs need to be fully labelled and carefully drawn to show significant points and behaviour at limits.
- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Both algebra and arithmetic can often be simplified using common factors and brackets.

General comments

Most candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of transformations. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

Candidates were able to employ standard methods of dealing with the roots of equations well.

(a) This was almost always correct.

(b) Most candidates factorised the expression correctly, however some made sign errors when substituting so did not achieve full credit.

(c) A minority of candidates realised that the previous two question parts had given them two of the coefficients needed and could write down the required equation immediately. Some candidates attempted to use a substitution of $w = z^2$ however this was unsuccessful. Many candidates did not attempt the question.

(d) Most candidates were able to form and solve the equation for p correctly.

Question 2

The general structure of a proof by induction was well understood by most candidates and there were some excellent solutions. There is however a need for more care in stating the hypothesis. In this case candidates needed to write down the function for k and also make the assumption that it can be divided by 74. The inductive step was attempted by rearrangement or by considering the difference between $f(k+1)$ and a multiple of $f(k)$. Many candidates did not show that one of the expressions is a multiple of 74. The best solutions took out a factor of 2 from 6^{4k} and from 38^k. Candidates are reminded that when not considering $f(k+1)$ directly they need to state why the result they have found implies that $f(k+1)$ is divisible by 74.
The final statement usually contained the required reference to $6^{4k} + 38^k – 2$ being divisible by 74 for all positive integers n.

Question 3

(a) The first two marks were almost universally achieved. The most elegant solutions kept the expressions fully factorised, factoring out the one twelfth and the initial $N(N+1)$. Those who multiplied out the brackets found themselves with a cubic or quartic requiring further factorisation.

(b) Nearly all candidates found the correct partial fractions. Those who set out the telescoping clearly did best. Candidates who wrote out the first few terms to show the effect of the powers of $\frac{1}{4}$ where generally successful when writing down the answer with the correct form for the final term.

(c) This was usually correct if **part (b)** had been answered correctly.

Question 4

(a) Most candidates correctly identified the transformations as stretch and rotation and gave them in the correct order. Candidates usually correctly described both the direction and scale factor for the one-way stretch. Some did not gain full credit as they did not describe both the angle and the centre of rotation for the second transformation.

(b) Most candidates clearly know how to find the inverse of a 2×2 matrix and could write the two matrices in the appropriate order. A minority of candidates did not divide by 14, however the most common error was to find $\mathbf{M}^{-1}$ as a single matrix.

(c) Most candidates calculated **M** accurately and made it clear that they were looking for invariant lines rather than points. There were many fully correct solutions.

(d) Responses to this question were almost always correct, with the method appearing to be well known to candidates.

Question 5

The majority of candidates demonstrated good knowledge and application of the required vector formulae. Several different methods were used to great effect, and it was evident that most students were comfortable with this topic. Candidates are advised to check cross products carefully; accuracy was often lost because of wrong signs.

(a) The cross product method was usually applied correctly.

(b) There were two methods which were most efficient in this question. The first was to find a vector joining D to a point of the plane and project it on to the normal direction. The second was to substitute the coordinates of D into the modified equation of the plane. Candidates who tried to find the base of the perpendicular from D to the plane often made errors in their working.

(c) There were many efficient and accurate solutions using the standard method. A handful of candidates tried first to find a point of intersection.

Question 6

(a) Almost all candidates wrote down the correct vertical asymptote. The equation of the oblique asymptote was often given correctly, although errors in the remainder were common when long division was used. Those who used the method of finding coefficients also commonly made errors, some of which were caused by the unknown a in the equation of the curve.

(b) Most candidates correctly differentiated the equation for C by using the quotient rule. Candidates generally then used the discriminant to explain that there were no real roots, however not all used the necessary condition $2a > 5$ to make their explanation convincing. An elegant solution was to rearrange to give $\frac{dy}{dx} = 1 + \frac{2a-5}{(x+2)^2}$ and to explain that this means $\frac{dy}{dx} > 1$.

(c) Most candidates remembered to label the asymptotes. Those who used the fact that there are no turning points drew two branches on the correct side of the asymptotes. A number of graphs showed one or two turning points because they were in the incorrect section of the *x*-*y* plane.

(d) (i) The idea of reflecting the graph in the *x* axis is well understood and many graphs correctly showed a cusp or "sharp bounce" off the x-axis and correct behaviour at the vertical asymptote.

(ii) Of the candidates who attempted this part, most were correct. However, many candidates did not attempt to draw the line.

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(iii) Most of those who got **part d(ii)** correct used this to write down equations to find critical points. Candidates connected these equations to the information given and therefore found the solution quickly by substituting $x = 3$ or $x = -3$ into their equation. Those who tried to work with inequalities were less successful.

Question 7

(a) Most candidates were able to produce an acceptable graph. The biggest problem was with the coordinate of the point furthest from the pole: many candidates forgot to take the square root or did not give correct polar form.

(b) The majority of candidates obtained the first three marks by writing down the correct integral and using integration by parts. When using a substitution, candidates are strongly advised to change all parts of the function, the limits and the $d\theta$ at the same time to avoid problems with constants and signs. When faced with $\int \frac{u^2}{1+u^2}\,\mathrm{d}u$ many tried a logarithmic expression or reversed their integration by parts, without success.

Those who formed the equation $\frac{u^2}{1+u^2} = 1 - \frac{1}{1+u^2}$ produced the best solutions. Another effective method was to use a second substitution of $U = \tan w$.

(c) Most candidates gained the first mark for using the correct function, and the last mark for establishing the change of sign. The differentiation was challenging and required both the chain rule and the product rule for three terms. The most common problems were the omission of one of the terms and sign errors.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics 12

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(a) The cross product method was usually applied correctly.

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FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics 13

Key messages

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Both algebra and arithmetic can often be simplified by the use of common factors and brackets.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. Candidates had opportunities to demonstrate some elegant approaches.

Comments on specific questions

Question 1

(a) Almost all candidates were able to form the correct determinant of the matrix.

Showing that it was not singular proved more challenging. A common approach was to use $k^2 \geqslant 0$ to show that the determinant cannot be zero. Answers needed to justify the statement det(A) $\neq 0$ which was not sufficient on its own. The alternative was to show that the equation $-5k^2 - 2 = 0$ has no solutions by considering the discriminant.

(b) There was a wide variety of methods used to solve this question. Some candidates recognised that the product of the determinants of A and A^{-1} is equal to 1 and achieved the answer quickly. The next most efficient method was to evaluate AA^{-1} and equate it to the identity matrix. This also gave a quick solution, although candidates did not, in fact, need to evaluate all elements of the product to be able to find the answer. Another possibility was to find the inverse of A^{-1} and compare it with the given matrix. The majority of candidates expended a lot of effort in calculating the inverse of the matrix A in terms of k and then equating elements with the given inverse. In this case many used an entry which gave a quadratic equation and did not check which was the correct solution. The solution $k = 0$ was often discarded.

Question 2

(a) Most candidates successfully used substitution and knew that the terms involving surds needed to be isolated before squaring. With many expansions required, it was impressive that the majority of candidates had completely correct working to get the required equation.

Very few candidates attempted to find the coefficients using the connecting formulae, and these solutions often involved slips in arithmetic.

(b) The standard method was usually correctly applied here.

(c) Most candidates used the method of summing the equations, remembered to use $S_0 = 3$, and found the correct answer. A few candidates used the formula connecting sums and products of roots of their equation from part **(a)** and the extra arithmetic gave more chance for error.

Question 3

(a) Most knew the terms 'stretch' and 'shear' and were able to correctly identify the order of the transformations. They gave full details to describe the stretch. Some candidates could explain the shear completely, using the words '*x* axis fixed' with '(0,1) mapped to (2,1)', for example.

(b) The concept of finding invariant lines was well understood. There were only a few candidates who were finding invariant points. The most common error was to miss one of the invariant lines either by cancelling before reaching the quadratic expression or obtaining $m = 0$ as one root but discarding it.

(c) This part was answered correctly by the majority of candidates.

Question 4

(a) Most candidates performed well on this proof by induction question. To establish the base case, candidates are reminded of the need to clearly (and separately) evaluate the left and right hand sides of the identity. The algebra for the inductive step was usually correct and showed enough detail to be convincing. The conclusion should include a statement of what they have proved. The words 'for all positive integers' were sometimes missing from the conclusion and incorrectly included in the hypothesis.

(b) The method of differences was well known with only a few cases of not enough detail being shown. The standard results for sums were applied correctly. The algebra required to reach the given answer proved challenging. The best solutions noticed at the start that $(2n + 1)$ was a common factor and then used long division or clear and systematic factorisation. Those who expanded $(2n + 1)^5$ gave themselves a much more complex expression to factorise. Several wrote down a quartic expression and went straight to the given answer with no method shown.

(c) There were many correct answers from candidates who realised that they needed to work with the coefficient of the highest power of *n* in the given answer for part **(b)**.

Question 5

The basic methods for vector questions are clearly well understood and many of the errors in this question were numerical. Candidates are advised to check they have written down the numbers accurately and that any cross products are correct, as errors quickly change the nature of the question.

(a) This part was well answered by most candidates. A common approach used the efficient method of finding a vector joining a point of one line to a point of the other line, and then taking the scalar product with the unit common normal. A few used the method of finding the points where the common perpendicular meets the two lines and the distance between them.

(b) This part was usually fully correct. Many candidates realised the appropriate normal vector had already been found in part **(a)** of the question and used it correctly.

(c) Candidates who realised that they needed two vectors within the plane were usually correct in finding a simple and brief solution. Many tried finding the cross product of the normals to the two planes, stating that this was a multiple of the direction of the line, before using the given points to find the values of *a*, *b*, *c* and *d*. There were some complete solutions but often candidates became lost in solving the equations. A common mistake was to assume that the normal vector of the required plane was perpendicular to both the given plane and the line of intersection. Candidates may find it helpful to draw a simple sketch.

Question 6

(a) Most candidates performed well on this part. They used the fact that $x^2 + 3 = 0$ has no real roots and so there are no vertical asymptotes and could write down the horizontal asymptote.

(b) This part was well answered, with only a few slips in arithmetic.

(c) Many curves seen were smooth and showed both maximum and minimum values and good approaches to the x axis as an asymptote. The better curves used a larger scale for y values to exaggerate the shape.

(d) There were some very good representations of the graph of $y^2 = \dfrac{x+1}{x^2+3}$.

Better responses recognised that the graph took both positive and negative values and had no part for $x < -1$. Many candidates did not include the part of the graph below the x axis. This meant they could not write down the coordinates of all the intersections and stationary points, although most could identify the positive ones using their previous results. The very best solutions recognised that the graph was vertical at the point $(-1, 0)$.

Question 7

(a) There were many good sketch graphs, showing a smooth loop in the first quadrant and its reflection in $\theta = \dfrac{\pi}{2}$. Candidates are advised to check that the behaviour at the pole is correctly shown, and the equation of the line of symmetry needs to be clearly identified. A few sketches showed incorrect extra lines.

(b) The most successful approach was to first eliminate θ by use of the double angle formula together with $x = r\cos\theta, y = r\sin\theta$. Having reached an equation connecting x and y some candidates spent time trying to rearrange the formula. This was not needed.

(c) Almost all candidates could write down the integral they needed to find the area. The best solutions expressed this as $\int_0^{\pi} \sin\theta\cos^2\theta d\theta$ and recognised that this is $-\left(\dfrac{1}{3}\cos^3\theta\right)_0^{\pi}$ to give the final answer. Candidates who use a substitution rather than recognition are advised to make sure that they change all of the function, the limits and the $\mathrm{d}\theta$, to ensure that they consider all factors and do not make sign errors. Those who tried using integration by parts needed to perform the process twice to get to the answer, but usually abandoned the work too soon. Methods involving various trigonometric identities were usually unsuccessful. Candidates are reminded that the accuracy marks must follow a correct method seen.

(d) Most candidates realised that they should be maximising r and knew that they needed to make r the subject prior to differentiation, or that they should justify why differentiating r^2 gives the same result. There were several different ways of expressing r but the differentiation always needed use of the product and chain rules. There were many good attempts, but the differentiation of a term such as $(\cos\theta)^{\frac{1}{2}}$ caused many problems. Errors in differentiation were often seen through missing constants and sign errors.

Those candidates who had gained a method mark for their attempt at differentiation of a correct expression could then go on to find the maximum value for r. Most of them showed correct use of relevant trigonometric identities to find a value for θ and hence for r. There were a pleasing number of fully correct solutions.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics 21

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when an answer is required in a certain form or in terms of a given variable.
- Candidates should make use of results derived or given in earlier parts of a question or given in the list of formulae (MF19).

General comments

Most candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that generally candidates were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers, particularly where answers were given within the question.

Comments on specific questions

Question 1

Good candidates showed clear working, starting from z^3 with the correct argument and listing all three roots in the required form. A significant number of candidates used $-\frac{1}{6}\pi$ as the argument of z^3 instead of $\frac{5}{6}\pi$. This led to incorrect answers when they attempted to take the cube root. A simple diagram would have aided in visualising that the argument is in the second quadrant, not the fourth.

Question 2

Most candidates differentiated the function twice, then applied the general formula to find the Maclaurin's series. Among the few who attempted to use the existing Maclaurin's series given in the list of formulae, many did this incorrectly by replacing x in the series with $1 + x^2$.

Question 3

(a) The majority of candidates found the first derivative correctly using parametric differentiation, although sign errors occasionally occurred when deriving the given answer.

(b) The attempts to find the second derivative varied in length, with strong candidates showing the required level of algebraic fluency and remembering to divide by $\frac{dx}{dt}$ after differentiating with respect to t.

Question 4

(a) Most candidates used integration by parts when attempting to find the reduction formula. Good candidates separated the integrand correctly, recognising that the derivative of $\tanh x$ is $\operatorname{sech}^2 x$ and using the hyperbolic identity relating tanh and sech.

(b) This part was well done with most candidates accurately applying the reduction formula, substituting I_2 to find I_4.

Question 5

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and good candidates applied the standard results for $\sum r$ and $\sum r^2$ to accurately derive the given result.

(b) Good candidates correctly adapted their solution to **(a)** and derived a suitable lower bound. There were some difficulties when simplifying the algebraic expressions which involved fractions.

(c) Most candidates showed clearly that the difference between U_2 and L_n is proportional to $\frac{1}{n}$, hence justifying the given limit. Some candidates chose a faster approach by taking the limit of the bounds separately and then finding the difference of the two limits.

Question 6

(a) Almost all candidates correctly substituted in sinh and cosh in terms of exponentials and worked clearly from the left-hand side of the equation to the right-hand side, fully justifying the given result.

(b) Most candidates took the correct approach to this question and completed it to a high standard. There was some inaccuracy when differentiating and comparing both sides of the equation and some problems with notation. A few candidates gave expressions instead of equations as their answer.

Question 7

(a) Almost all candidates applied the given substitution correctly. A few did not express their answer in terms of x.

(b) The majority of candidates divided through by x and then arrived at the correct integrating factor. Good candidates fully simplified the right-hand side of the equation after multiplying by the integrating factor. A few candidates struggled to express the left-hand side of the equation as a derivative of the product of the integrating factor and the function. Integration by parts led to the integral in part **(a)**. A frequently seen error was omitting to multiply the constant term by the integrating factor, especially when this was done before evaluating the constant.

Question 8

(a) The majority of candidates found the correct expression for the determinant of the corresponding matrix in terms of a. Some candidates then set this determinant not equal to zero to generate a unique solution.

(b) A minority of candidates were able to formulate a complete description, distinguishing between the cases $a \neq -3$ and $a = -3$ and fully justifying their conclusion. On occasion, expressions were divided through by a term which could be zero, rather than this term being factorised out.

(c) Candidates who used the vector product method to find the eigenvectors tended to be most successful, although sign errors were common. Some responses gave eigenvectors which did not have the correct properties, which could have been checked by performing matrix multiplication. Almost all showed an awareness of how to find the matrices **P** and **D**. A few did not perform the full number of operations on the eigenvalues of **A** to form **D**.

(d) Good candidates were able to maintain accuracy throughout their solution, both when substituting into the characteristic equation and when making $14\mathbf{A} + 24\mathbf{I}$ the subject before squaring both sides.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics 22

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FURTHER MATHEMATICS

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Further Pure Mathematics 23

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General comments

Most candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. Sometimes candidates did not fully justify their answers, particularly where answers were given within the question. There were many scripts of a very high standard.

Comments on specific questions

Question 1

Most candidates realised that they needed to complete the square and were usually successful. A few other less efficient methods were seen involving elaborate changes of variable. A small minority of candidates lost the final mark due to not evaluating the inverse sine expression.

Question 2

(a) The majority of candidates gained full marks for this part, being familiar with the required formula for arc length and the appropriate hyperbolic identity.

(b) Good candidates maintained accuracy when differentiating and completed the question fully by then integrating the Maclaurin's series. A few candidates attempted to use the standard Maclaurin's series from the list of formulae. This approach was rarely successfully due to the square root being dealt with incorrectly.

Question 3

(a) The majority of candidates accurately differentiated both sides of the equation implicitly and showed enough working to justify the given answer. In particular, the substitution of the values needed to be demonstrated.

(b) Good candidates accurately used implicit differentiation again to find an equation involving the second derivative. The most common error was in obtaining $24y^2\dfrac{dy}{dx}$ rather than $24y^2\left(\dfrac{dy}{dx}\right)^2$.

Many candidates used the quotient rule on their rearranged expression from part (a) and, although this required more algebraic manipulation, was usually also successful.

Question 4

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles and compared with a suitable integral with appropriate limits to accurately derive the given result.

(b) Good candidates correctly adapted their solution to **(a)** and derived a suitable upper bound.

(c) Candidates who had suitable bounds in terms of $\frac{1}{N}$ were able to take the limit as $N \to \infty$, clearly identifying the lower and upper bounds.

Question 5

(a) Almost all candidates approached this question and completed it to a high standard. There were some inaccuracies when comparing coefficients to find the particular integral and some problems with notation. A few candidates gave expressions instead of equations as their answer.

(b) Most candidates correctly used their particular integral from part **(a)**. There were also some problems with notation for this part, with a few candidates using an arrow instead of an equals sign. Most found $R = 13$ successfully, however sometimes the numerator and denominator had been reversed or ϕ had not been evaluated.

Question 6

(a) The majority of candidates used the formula for the sum of a geometric progression correctly. Good candidates showed their working clearly when dividing the numerator and denominator by z^2 to fully justify the given answer.

(b) Almost all knew that de Moivre's theorem related the series to the geometric progression in part **(a)**. Strong candidates accurately took the imaginary part, after simplifying the numerator and denominator, which led to the given answer. Often, crucial steps were omitted, and therefore answers were not full and detailed enough.

Question 7

(a) The differentiation was done well.

A common error was to obtain $\frac{9}{2} \times \frac{1}{\sqrt{\frac{x^2}{9}-1}}$ rather than the correct $\frac{9}{2} \times \frac{1}{3} \frac{1}{\sqrt{\frac{x^2}{9}-1}}$

The simplification was found to be more challenging, especially amongst those who converted the inverse cosh to logarithmic form before differentiating.

(b) The majority of candidates divided through by x and then arrived at the correct integrating factor. Good candidates fully simplified the right-hand side of the equation after multiplying by the integrating factor. A frequently seen error was failing to multiply the constant term by the integrating factor, especially when this was done before evaluating the constant.

Question 8

(a) Strong candidates recalled how to form the cartesian equation of a plane given a direction vector perpendicular to the plane and a point on the plane.

(b) Candidates generally showed good understanding of the vector equation of a line. Most candidates used the components successfully to prove the required result, however a few did not include z = 0.

(c) This was a given answer so complete justification was necessary. A few candidates did not expand fully when deriving the characteristic equation or neglected to complete the question by finding the roots of the cubic equation.

(d) Candidates who used the vector product method to find the eigenvectors tended to be most successful, although sign errors were common. Some responses gave eigenvectors which did not have the correct properties, which could have been checked by performing matrix multiplication. Almost all candidates showed an awareness of how to find the matrices **P** and **D**. A small number of candidates forgot to include the power *n* or used a zero vector in their final answer.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics 31

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working so that the offered solution is communicated clearly and completely. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in case a diagram is provided, to annotated it. This helps understand the problem and model it correctly. For example, in **Question 5**, the candidates who drew a diagram realised that, while the tensions on the particles were in opposite directions, both frictions were directed towards the centre of the turntable. As a result these candidates were typically able to write the correct equation for the equilibrium of forces.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's 2nd Law, for example to set up a differential equation, or in questions involving collisions, they must ensure they explicitly mention the mass, or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of elementary algebra, as in **Question 7** part **(a)**.

Comments on specific questions

Question 1

This part question was answered correctly by many candidates. Some of them chose to represent the horizontal component of sphere A after the collision as $v_A \cos\alpha$ instead of v_A and, even though this choice was correct, it often led to errors as they had to solve a system of equations in v_B and $v_A \cos\alpha$. The candidates who realised that the energy of sphere A after the collision included both components often managed to obtain the correct answer, showing good algebraic manipulative skills. Errors seen included omitting the masses in the equation of the principle of conservation of linear momentum, or writing the mass of sphere B as m and not $5m$.

Question 2

(a) The candidates who drew a diagram understood that they had to consider the vertical component of the tensions in the equation for the equilibrium of forces, and often wrote a correct equation. A common error was to consider only the tension in one half of the string. Some candidates did not apply Hooke's law correctly; if one applied the law to one half of the string only then the value of the natural length of the string also had to be halved.

(b) To answer this part question, the candidates had to apply the principle of conservation of mechanical energy. Most candidates realised that, at point M, the particle has no elastic potential energy and so the equation had only three terms, one per type of energy (elastic potential, gravitational potential, and kinetic). Some candidates did not correctly identify the value of the initial extension of the string.

Question 3

Some candidates found this question challenging. A common error was to use distances, instead of velocities, to describe the direction of motion. The candidates who realised that the direction of motion was given by the ratio of the components of the velocity vector were usually able to correctly model the problem and answer the question. They did this using a variety of approaches, including the formula for the product of gradients of perpendicular lines, and the scalar product of the components of the velocity vectors. The most elegant answers used inverse tangents.

Question 4

(a) Many candidates scored the first method mark as they correctly resolved forces parallel to the inclined plane, but then were not able to find a second suitable equation to eliminate the friction (e.g., by calculating moments about point O). Some candidates resolved forces perpendicular to the surface, but in doing so introduced a new variable (normal reaction or friction) that they were then unable to eliminate. Other candidates opted for the equilibrium of vertical and horizontal forces, but were typically not successful in proceeding further. Stronger responses calculated the moments about the point where the ring touches the plane.

(b) The candidates who could answer the previous part question correctly typically went on to find the correct solution to this part. They realised that they could use the equation for the equilibrium of the forces perpendicular to the surface, together with another suitable equation, e.g., the equation of moments about point O.

Question 5

(a) The candidates who drew a diagram realised that while the tensions on the particles had opposite directions, both frictions were directed towards the centre of the turntable. They usually had no problems writing the equation for Newton's Law applied to particle A and to obtain the correct answer. Some candidates attempted to equate expressions for the tensions for particle A and particle B; in doing so they introduced an additional unknown (k), and so provided an answer in terms of m, g, and k and not in terms of only m and g, as requested.

(b) Most of the candidates who answered part **(a)** correctly had no problems scoring full marks in this part question. A common error was to use the mass of particle A in the expression of the acceleration.

Question 6

(a) This part question was answered well by many candidates, who showed a good understanding of how to set up and successfully solve a differential equation, including the use of boundary conditions. Some candidates did not separate the variables correctly or differentiated the function instead of integrating it. The correct answer could be expressed in different forms, e.g., $\frac{1}{2}+\frac{5e^t}{3e^t-1}$, $\frac{1}{2}+\frac{5}{3-e^{-t}}$, $\frac{13e^t-1}{6e^t-2}$, $\frac{13-e^{-t}}{6-2e^{-t}}$.

(b) This part question was also answered well, even though it proved more challenging than part **(a)**. The candidates who provided their answers to part **(a)** in the form $\frac{1}{2}+\frac{5e^t}{3e^t-1}$, could integrate this function directly, and usually did so well. Those instead whose answer was in the form $\frac{1}{2}+\frac{5}{3-e^{-t}}$ used two strategies to integrate the second term: multiply numerator and denominator by e^t (thus obtaining $\frac{1}{2}+\frac{5e^{t}}{3e^{t}-1}$, which they usually integrated easily), or apply the substitution $e^{t} = u$. In the latter case they then had to use partial fractions on their integrand function, and only a few managed to do so correctly. Most of the candidates who had to integrate $\frac{13e^{t}-1}{6e^{t}-2}$ or $\frac{13-e^{-t}}{6-2e^{-t}}$ had little success in reaching the correct solution. The most common error was to integrate a function of the form $\frac{A}{Be^{\pm t}\pm C}$ into $A\ln(Be^{\pm t}\pm C)$.

Question 7

(a) This part question was answered well by many candidates, who showed a good understanding of the use of the principle of the conservation of mechanical energy, together with Newton's second law. The most common error was to miscalculate the change in gravitational potential energy.

(b) The key to answer this question was to ignore the horizontal component of the velocity and to focus only on the vertical component, starting from moment the particle loses contacts with the sphere. Only the strongest candidates realised this. Some responses included calculations for the initial velocity of the particle, even though there was no use for it.

(c) This last part question was answered using a variety of approaches. Many candidates rearranged the formula $V = v + gt$, where V and v are the vertical components of the velocity of the particle just before it hits the ground (V) and when it leaves the sphere (v), other candidates used the formula $s = \frac{1}{2}(v + V)t$ where s is the vertical distance travelled by the particle while free falling.

Finally, a few candidates opted for the formula $s = vt + \frac{1}{2}gt^2$. Only the best responses used the vertical component of the velocities.

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics 32

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working so that the offered solution is communicated clearly and completely. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram or, in case a diagram is provided, to annotated it. This helps understand the problem and model it correctly. For example, in **Question 5**, the candidates who drew a diagram realised that, while the tensions on the particles were in opposite directions, both frictions were directed towards the centre of the turntable. As a result these candidates were typically able to write the correct equation for the equilibrium of forces.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's 2nd Law, for example to set up a differential equation, or in questions involving collisions, they must ensure they explicitly mention the mass, or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of elementary algebra, as in **Question 7** part **(a)**.

Comments on specific questions

Question 1

This part question was answered correctly by many candidates. Some of them chose to represent the horizontal component of sphere A after the collision as $v_A \cos\alpha$ instead of v_A and, even though this choice was correct, it often led to errors as they had to solve a system of equations in v_B and $v_A \cos\alpha$. The candidates who realised that the energy of sphere A after the collision included both components often managed to obtain the correct answer, showing good algebraic manipulative skills. Errors seen included omitting the masses in the equation of the principle of conservation of linear momentum, or writing the mass of sphere B as m and not $5m$.

Question 2

(a) The candidates who drew a diagram understood that they had to consider the vertical component of the tensions in the equation for the equilibrium of forces, and often wrote a correct equation. A common error was to consider only the tension in one half of the string. Some candidates did not apply Hooke's law correctly; if one applied the law to one half of the string only then the value of the natural length of the string also had to be halved.

(b) To answer this part question, the candidates had to apply the principle of conservation of mechanical energy. Most candidates realised that, at point M, the particle has no elastic potential energy and so the equation had only three terms, one per type of energy (elastic potential, gravitational potential, and kinetic). Some candidates did not correctly identify the value of the initial extension of the string.

Question 3

Some candidates found this question challenging. A common error was to use distances, instead of velocities, to describe the direction of motion. The candidates who realised that the direction of motion was given by the ratio of the components of the velocity vector were usually able to correctly model the problem and answer the question. They did this using a variety of approaches, including the formula for the product of gradients of perpendicular lines, and the scalar product of the components of the velocity vectors. The most elegant answers used inverse tangents.

Question 4

(a) Many candidates scored the first method mark as they correctly resolved forces parallel to the inclined plane, but then were not able to find a second suitable equation to eliminate the friction (e.g., by calculating moments about point O). Some candidates resolved forces perpendicular to the surface, but in doing so introduced a new variable (normal reaction or friction) that they were then unable to eliminate. Other candidates opted for the equilibrium of vertical and horizontal forces, but were typically not successful in proceeding further. Stronger responses calculated the moments about the point where the ring touches the plane.

(b) The candidates who could answer the previous part question correctly typically went on to find the correct solution to this part. They realised that they could use the equation for the equilibrium of the forces perpendicular to the surface, together with another suitable equation, e.g., the equation of moments about point O.

Question 5

(a) The candidates who drew a diagram realised that while the tensions on the particles had opposite directions, both frictions were directed towards the centre of the turntable. They usually had no problems writing the equation for Newton's Law applied to particle A and to obtain the correct answer. Some candidates attempted to equate expressions for the tensions for particle A and particle B; in doing so they introduced an additional unknown (k), and so provided an answer in terms of m, g, and k and not in terms of only m and g, as requested.

(b) Most of the candidates who answered part **(a)** correctly had no problems scoring full marks in this part question. A common error was to use the mass of particle A in the expression of the acceleration.

Question 6

(a) This part question was answered well by many candidates, who showed a good understanding of how to set up and successfully solve a differential equation, including the use of boundary conditions. Some candidates did not separate the variables correctly or differentiated the function instead of integrating it. The correct answer could be expressed in different forms, e.g., $\frac{1}{2}+\frac{5e^t}{3e^t-1}$, $\frac{1}{2}+\frac{5}{3-e^{-t}}$, $\frac{13e^t-1}{6e^t-2}$, $\frac{13-e^{-t}}{6-2e^{-t}}$.

(b) This part question was also answered well, even though it proved more challenging than part **(a)**. The candidates who provided their answers to part **(a)** in the form $\frac{1}{2}+\frac{5e^t}{3e^t-1}$, could integrate this function directly, and usually did so well. Those instead whose answer was in the form $\frac{1}{2}+\frac{5}{3-e^{-t}}$ used two strategies to integrate the second term: multiply numerator and denominator by e^t (thus obtaining $\frac{1}{2}+\frac{5e^t}{3e^t-1}$, which they usually integrated easily), or apply the substitution $e^t = u$. In the latter case they then had to use partial fractions on their integrand function, and only a few managed to do so correctly. Most of the candidates who had to integrate $\frac{13e^t-1}{6e^t-2}$ or $\frac{13-e^{-t}}{6-2e^{-t}}$ had little success in reaching the correct solution. The most common error was to integrate a function of the form $\frac{A}{Be^{\pm t}\pm C}$ into $A\ln(Be^{\pm t}\pm C)$.

Question 7

(a) This part question was answered well by many candidates, who showed a good understanding of the use of the principle of the conservation of mechanical energy, together with Newton's second law. The most common error was to miscalculate the change in gravitational potential energy.

(b) The key to answer this question was to ignore the horizontal component of the velocity and to focus only on the vertical component, starting from moment the particle loses contacts with the sphere. Only the strongest candidates realised this. Some responses included calculations for the initial velocity of the particle, even though there was no use for it.

(c) This last part question was answered using a variety of approaches. Many candidates rearranged the formula $V = v + gt$, where V and v are the vertical components of the velocity of the particle just before it hits the ground (V) and when it leaves the sphere (v), other candidates used the formula $s = \frac{1}{2}(v + V)t$ where s is the vertical distance travelled by the particle while free falling.

Finally, a few candidates opted for the formula $s = vt + \frac{1}{2}gt^2$. Only the best responses used the vertical component of the velocities.

FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics 33

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working so that the offered solution is communicated clearly and completely. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

General comments

In most questions the majority of candidates understood what method to use, however some omitted to draw a suitable diagram, or to annotate the given diagram, and this resulted in writing incorrect equations.

Candidates are reminded that, when the answer is given, as in **Questions 3(a)** and **6(a)**, they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Most candidates made a good attempt at this question. As always in questions on this topic, equations resulting from the conservation of linear momentum and Newton's law of restitution are required. The first of these enabled a value for $\cos\theta$ to be found and the second led to a value for the coefficient of restitution. The errors that occurred were usually sign errors in one or both of the equations.

Question 2

(a) Some concise and fully correct solutions were seen in this question. Most candidates were able to write down at least some of the relevant equations but were not able to combine these to find the extension in the string. An expression of Hooke's law $T = \frac{2mgx}{a}$ was usually present together with an application of Newton's second law horizontally, $T = \frac{mv^2}{r}$, with $v^2 = \frac{1}{2}ga$. The two expressions for the tension T are then equated. Many candidates were unable to make any meaningful progress from this point. Better responses then showed r in terms of a and either the extension of the string x or the extended length l of the string. This results in a quadratic equation from which the extension can be found in terms of a.

Some candidates resolved vertically to obtain $T = kmg$ and attempted to combine this with their other equations. Although it is possible for this approach to lead to a quadratic equation in k, and from there to find the extension in terms of a, very few candidates were able to negotiate this more difficult path successfully.

(b) The vertical resolution equation $T = kmg$ was required in this part, and when combined with the Hooke's law expression and the result of **part (a)** the value of k can be determined.

Question 3

(a) Most candidates were able to write down an energy equation and apply Newton's second law twice, at the initial position of the particle and at the point where the string goes slack. These three equations are then combined to eliminate the speeds in the two positions and give the required expression involving *S* and *T*. Most candidates were able to obtain the given expression, but the final accuracy mark was only awarded when the candidates gave a convincing solution. It is important to remember that in 'show that' questions each step in the working must be clearly shown.

(b) This part was answered well by the majority of candidates.

Question 4

There were some excellent fully correct solutions to this problem, and most candidates recognised that the method required them to form two energy equations each involving elastic potential energy, gravitational potential energy and kinetic energy and they made a reasonable attempt at doing so. There were several common errors. Some candidates used an incorrect formula for elastic potential energy, usually $\frac{\lambda x}{2l}$ instead of the correct formula $\frac{\lambda x^2}{2l}$. This leads to energy equations which are dimensionally incorrect. Candidates need to be aware that it is important on this paper that any equation is dimensionally correct, and they should check this aspect when they have written down any equation.

A second common misconception was the assumption that at *C*, when the spring is compressed, the elastic potential energy is zero. A less common error was to equate the elastic potential energy at a point to the sum of the kinetic and gravitational energies at that point.

Candidates who formed their two energy equations with minor inaccuracies usually proceeded to eliminate *V* and find a value for *k*.

Question 5

(a) Almost all candidates knew how to answer this question, and most did so accurately. The common method was to write down the area and distance of the centre of mass from *OC* for each of the triangles *OBC* and *OAC*. A moments equation was then formed and simplified to give the distance of the centre of mass of triangle *ABC* from *OC*. The errors that occurred were usually in the distances of the centre of mass of each of the triangles from *OC* or, less often, in the algebraic manipulation of the moments equation.

Some candidates also used a similar method to find the distance of the centre of mass from *OB*.

This was not required, and incidentally could be written down as equal to $\frac{1}{3} \times 18a = 6a$.

A minority of candidates used the simpler method of solving the problem by considering the system as equivalent to particles at points with coordinates (0, 18*a*), (*x*, 0) and (24*a*, 0) giving the x-coordinate of the centre of mass as $\frac{1}{3}(x + 24a)$.

(b) This part proved to be challenging for many candidates who were unable to negotiate the geometry of the situation to find a correct expression for $\tan\theta$. The common error was to write $\tan\theta = \frac{\bar{y}}{\bar{x}}$, presumably because this is often the case in problems on this topic. The simplest correct expression is $\tan\theta = \frac{18a - 6a}{\bar{x}}$, although there are other equivalent expressions such as is $\tan\theta = \frac{\bar{y}}{15a - \bar{x}}$. These alternative expressions often involve the need to find an expression for $\bar{y}$.

Question 6

(a) Candidates answered this part well. Some candidates did not show sufficient working, essential in a 'show that' request.

(b) There are at least three different methods for solving this problem. Most candidates opted to write down an expression for $\tan\alpha$, where α is the angle that the direction of motion after 5 seconds makes with the horizontal. The condition that this direction is perpendicular to the initial direction of motion, $\tan\alpha\tan\theta = -1$, is then used together with the result from **part (a)** to find the values of u and $\sin\theta$. A common error in this approach was to use $\tan\alpha\tan\theta = +1$.

Another method is to consider the horizontal component of the initial velocity, together with the condition that the velocity after 5 seconds is perpendicular to the initial velocity. This leads to $u\cos\theta = \frac{3}{4}u\sin\theta$ and the value of $\tan\theta$. The value of u can then be found by using the result from **part (a)**. Alternatively for this final step, the value for u can be found by considering the vertical components of the velocity initially and after 5 seconds.

It was common to see a sign error in this part, leading to an incorrect value for u either on its own or together with the correct value. This incorrect value, $\frac{1000}{7}$, corresponds to a point where the particle is still on its upward path. Those candidates who drew a diagram usually avoided this sign error.

Question 7

(a) Many candidates were able to set up the correct differential equation, solve it including the application of the initial condition, and find the required expression for v. Sometimes there were sign errors or algebraic errors, but most candidates obtained a logarithmic term on integration. A common error was the omission of a modulus sign in the logarithmic term. Other candidates included the modulus sign but then removed it incorrectly when rearranging to find an expression for v.

A few candidates attempted to solve the problem by using *suvat* equations, even though the acceleration was not constant.

(b) Most candidates were able to integrate their result from **part (a)** and use the given initial condition to find an expression for x.

(c) Many candidates answered this correctly.

FURTHER MATHEMATICS

Paper 9231/41
Further Probability and Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

Almost all candidates attempted all the questions. The standard was generally good, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one.

Comments on specific questions

Question 1

The majority of candidates answered this question well. The most common error was the use of a z value instead of a t value in the calculation of the confidence interval. A minority of candidates calculated s^2 by dividing by n instead of $n - 1$.

Candidates should be aware that their final answer for the confidence interval should be in the form of an interval, in this case (37.9, 47.7) rather than 42.8 ± 4.91.

Question 2

It was clear that candidates were generally not as well prepared for the sign test as for the other hypothesis tests in the paper. Whilst there were some fully correct solutions, many candidates used a Wilcoxon signed-rank test, presumably thinking that this was the sign test. Some candidates attempted to apply a parametric test.

As was common throughout the paper, candidates were often unable to give an appropriate conclusion to the test which was in context and with some level of uncertainty. The common error was to give a conclusion that was too assertive. Candidates should be aware that the outcome of a test *provides evidence* which is either *sufficient* or *insufficient* to reject H_0; the outcome never *proves* anything so care should be taken to avoid definite statements in conclusions. It is also important to be aware that it is the null hypothesis H_0 that is being tested, so it is H_0 that is rejected or accepted. In this case, a conclusion such as 'there is insufficient evidence to suggest that practical results are greater than written results' is required and not 'there is sufficient evidence to suggest that practical results are greater than written results'.

Question 3

(a) Most candidates were able to carry out the Wilcoxon signed-rank test accurately. The common errors that were seen were in the statement of the hypotheses and in the expression of the conclusion of the test. The hypotheses were often not stated in terms of the population parameter: they referred to the median, with the word 'population' omitted. Sometimes, the hypotheses referred to the (population) mean or simply used the symbol μ.

The other common error was to give a conclusion that was too assertive. The outcome of a test *provides evidence* which is either *sufficient* or *insufficient* to reject H_0; the outcome never *proves* anything so care should be taken to avoid definite statements in conclusions. As with other hypothesis tests, it is also important to be aware that it is the null hypothesis H_0 that is being tested, so it is H_0 that is rejected or accepted.

(b) Only a minority of candidates were able to provide an appropriate assumption that was required for the test in **part (a)** to be valid. A common error was to suggest that it was the data rather than the population or underlying distribution that had to be symmetrical about the (population) median. Many candidates made irrelevant comments about normality or randomness or independence.

Question 4

(a) This part was answered well with accurate differentiation, using either the product rule or the quotient rule.

(b) Most candidates answered this correctly.

(c) The most efficient way seen to find the value of $P(X = 4)$ was the use a binomial expansion of the expression for $G_X(t)$ before identifying the coefficient of t^4. Those candidates who did appreciate this were usually successful in obtaining the correct answer. Some candidates chose to find the fourth derivative of $G_X(t)$, evaluate it at $t = 0$ and then divide by 4!. This often led to algebraic and numerical errors. Other candidates believed that $P(X = 4)$ was equal to $G_X(4)$ which is not correct.

Question 5

(a) The majority of candidates stated appropriate hypotheses that referred to the quality of the brushes, independence and company. Candidates need to be aware that complete hypotheses are required; statements such as quality is independent, or, for the alternate hypothesis 'it is not independent', are not complete.

Almost all candidates found and showed the correct expected frequencies, with most going on to find the test statistic accurately. A small number made errors in the use of the formula, for example by having the observed values in the denominator. Usually, the test statistic was compared with the correct critical value of 5.991 and a correct decision made about H_0. Most candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language. As in other questions, sometimes the conclusion was too assertive; examples of such statements are '*sufficient* evidence for independence…' rather than the correct '*insufficient* evidence to reject independence…'.

(b) Most candidates recognised that it is the proportions of brushes in each category that is important and not the frequencies, and offered an appropriate comment.

Question 6

(a) Most candidates recognised this question as a paired sample *t*-test problem, signalled by a 'before and after' situation. A common error was in the statement of the hypotheses, omitting any reference to the 1 second reduction in times that was being tested. As in other questions, the conclusion was sometimes too assertive and sometimes incorrectly expressed.

A minority of candidates treated the question as an independent samples *t*-test problem instead of a paired sample *t*-test.

(b) Many candidates were not able to articulate clearly the assumption needed for a paired sample *t*-test. The key feature is that the *population differences* should be normally distributed and not the *population*. Some candidates referred to randomness or independence, or both.

Question 7

This question was answered well by many candidates.

(a) Most candidates formed the correct integrals for E(X) and E($\sqrt{X}$), evaluated them accurately and used their results in the formula for variance.

(b) Most candidates worked through this part methodically to obtain the probability density function for Y. Some candidates did not give the correct interval for y and some omitted '0 otherwise' giving an incomplete expression for the probability density function.

(c) Almost all candidates knew what they had to do to find the median value, but some candidates gave their final answer as a decimal rather than in exact form as required.

FURTHER MATHEMATICS

Paper 9231/42
Further Probability and Statistics 42

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working, so that there is no doubt that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

Almost all candidates attempted all the questions. The standard was generally good, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one.

Comments on specific questions

Question 1

The majority of candidates answered this question well. The most common error was the use of a z value instead of a t value in the calculation of the confidence interval. A minority of candidates calculated s^2 by dividing by n instead of $n - 1$.

Candidates should be aware that their final answer for the confidence interval should be in the form of an interval, in this case (37.9, 47.7) rather than 42.8 ± 4.91.

Question 2

It was clear that candidates were generally not as well prepared for the sign test as for the other hypothesis tests in the paper. Whilst there were some fully correct solutions, many candidates used a Wilcoxon signed-rank test, presumably thinking that this was the sign test. Some candidates attempted to apply a parametric test.

As was common throughout the paper, candidates were often unable to give an appropriate conclusion to the test which was in context and with some level of uncertainty. The common error was to give a conclusion that was too assertive. Candidates should be aware that the outcome of a test *provides evidence* which is either *sufficient* or *insufficient* to reject H_0; the outcome never *proves* anything so care should be taken to avoid definite statements in conclusions. It is also important to be aware that it is the null hypothesis H_0 that is being tested, so it is H_0 that is rejected or accepted. In this case, a conclusion such as 'there is insufficient evidence to suggest that practical results are greater than written results' is required and not 'there is sufficient evidence to suggest that practical results are greater than written results'.

Question 3

(a) Most candidates were able to carry out the Wilcoxon signed-rank test accurately. The common errors that were seen were in the statement of the hypotheses and in the expression of the conclusion of the test. The hypotheses were often not stated in terms of the population parameter: they referred to the median, with the word 'population' omitted. Sometimes, the hypotheses referred to the (population) mean or simply used the symbol μ.

The other common error was to give a conclusion that was too assertive. The outcome of a test *provides evidence* which is either *sufficient* or *insufficient* to reject H_0; the outcome never *proves* anything so care should be taken to avoid definite statements in conclusions. As with other hypothesis tests, it is also important to be aware that it is the null hypothesis H_0 that is being tested, so it is H_0 that is rejected or accepted.

(b) Only a minority of candidates were able to provide an appropriate assumption that was required for the test in **part (a)** to be valid. A common error was to suggest that it was the data rather than the population or underlying distribution that had to be symmetrical about the (population) median. Many candidates made irrelevant comments about normality or randomness or independence.

Question 4

(a) This part was answered well with accurate differentiation, using either the product rule or the quotient rule.

(b) Most candidates answered this correctly.

(c) The most efficient way seen to find the value of $P(X = 4)$ was the use a binomial expansion of the expression for $G_X(t)$ before identifying the coefficient of t^4. Those candidates who did appreciate this were usually successful in obtaining the correct answer. Some candidates chose to find the fourth derivative of $G_X(t)$, evaluate it at $t = 0$ and then divide by 4!. This often led to algebraic and numerical errors. Other candidates believed that $P(X = 4)$ was equal to $G_X(4)$ which is not correct.

Question 5

(a) The majority of candidates stated appropriate hypotheses that referred to the quality of the brushes, independence and company. Candidates need to be aware that complete hypotheses are required; statements such as quality is independent, or, for the alternate hypothesis 'it is not independent', are not complete.

Almost all candidates found and showed the correct expected frequencies, with most going on to find the test statistic accurately. A small number made errors in the use of the formula, for example by having the observed values in the denominator. Usually, the test statistic was compared with the correct critical value of 5.991 and a correct decision made about H_0. Most candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language. As in other questions, sometimes the conclusion was too assertive; examples of such statements are '*sufficient* evidence for independence…' rather than the correct '*insufficient* evidence to reject independence…'.

(b) Most candidates recognised that it is the proportions of brushes in each category that is important and not the frequencies, and offered an appropriate comment.

Question 6

(a) Most candidates recognised this question as a paired sample *t*-test problem, signalled by a 'before and after' situation. A common error was in the statement of the hypotheses, omitting any reference to the 1 second reduction in times that was being tested. As in other questions, the conclusion was sometimes too assertive and sometimes incorrectly expressed.

A minority of candidates treated the question as an independent samples *t*-test problem instead of a paired sample *t*-test.

(b) Many candidates were not able to articulate clearly the assumption needed for a paired sample *t*-test. The key feature is that the *population differences* should be normally distributed and not the *population*. Some candidates referred to randomness or independence, or both.

Question 7

This question was answered well by many candidates.

(a) Most candidates formed the correct integrals for E(X) and E($\sqrt{X}$), evaluated them accurately and used their results in the formula for variance.

(b) Most candidates worked through this part methodically to obtain the probability density function for Y. Some candidates did not give the correct interval for y and some omitted ‘0 otherwise’ giving an incomplete expression for the probability density function.

(c) Almost all candidates knew what they had to do to find the median value, but some candidates gave their final answer as a decimal rather than in exact form as required.

FURTHER MATHEMATICS

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Key messages

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Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, ‘there is insufficient evidence to support the claim that….’ rather than ‘the test proves that….’.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were commonly seen in **Question 2**.

Comments on specific questions

Question 1

Candidates need to be aware that hypotheses for parametric tests must be stated in terms of *population parameters*; a significant number of candidates simply referred to ‘median’ rather than ‘population median’. Others used the incorrect parameter μ which is the standard notation for population mean not population median.

The vast majority of candidates were able to rank the data appropriately from 1 to 19 and identify the correct test statistic of 70. The few who used reversed ranks to find a rank sum of 110 usually also then found the correct test statistic. A small number of candidates had 59 as the test statistic. This relates to m = 9 and n = 9 rather than n = 10. The appropriate decision was then usually made to accept H_0. However, the final mark could not always be awarded as candidates either used definite language (such as ‘prove’) or lacked language of uncertainty such as ‘insufficient evidence’. Some stated incorrectly that there was ‘sufficient evidence to accept H_0’, which is not the same as ‘insufficient evidence to reject H_0’.

A small number of candidates decided to use the normal approximation for R_m, given in the tables, but this is not appropriate when the values of m and n are in the table. A similarly small number chose the wrong test, attempting a Wilcoxon signed-rank test on the differences.

Question 2

(a) There were excellent solutions to this part, which were both accurate and set out clearly. There were also many solutions in which the method was not clear. Candidates would be well advised to present their working in stages, completing the calculations as they proceed, rather than combining formulae into a single formula before any evaluation. This latter approach often resulted in algebraic errors.

The value of Σx was usually found correctly, often by eliminating the sample mean from expressions for the two confidence limits rather than by using the fact that sample mean was midway between the limits. Most candidates were able to use an expression for confidence limits, usually with the correct t-value 2.718 (for $n - 1 = 11$ degrees of freedom). Some candidates used an incorrect t value, and others used a z value.

(b) The majority of candidates were able to correctly state that the population or the underlying distribution was assumed to be normal. Other candidates made statements that were too vague or incorrect such as ‘it is normally distributed’, ‘the population is symmetrical’, or ‘the data is normal’. A few candidates stated that the ‘population mean is normally distributed’, which is incorrect as population mean is a single value.

Question 3

The majority of candidates stated appropriate hypotheses that referred to the reliability of buses, independence and the bus company. Candidates need to be aware that complete hypotheses are required; statements such as ‘buses are independent, or, for the alternate hypothesis, ‘it is not independent’ are not complete.

Almost all candidates found and showed the correct expected frequencies, with most going on to find the test statistic accurately. A small number made errors in the use of the formula, for example, by having the observed values in the denominator. Usually, the test statistic was compared with the correct critical value of 9.488 and a correct decision made about H_0. Most candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language. Credit was not awarded for incorrect statements such as ‘*sufficient* evidence for independence…’ rather than the correct ‘*insufficient* evidence to reject independence…’.

Question 4

(a) Almost all candidates found the value of c correctly by setting $G_X(1) = 1$.

(b) Most candidates knew that they needed to find $G'_X(1)$ and that the most efficient way to do this was to use the product rule to differentiate their expression. The most common error was to differentiate both parts of the product at the same time, resulting in a single term. Some candidates opted to expand their expression for G before differentiating it term by term. This often introduced errors and was also unnecessarily time-consuming.

(c) Most candidates knew that they needed to square $G'_X(t)$ and then find the first and second differentials to use in the relevant formula for Var(Y). Most candidates used the product rule twice. Those who had expanded their probability generating function for X had more time-consuming differentiation again, and errors crept in. Very few candidates used the neat alternative method in this part, based on Var (Y) = 2Var(X).

(d) The majority of candidates were aware that P(Y = 5) is the coefficient of t^5 in the expansion of the probability generating function for Y and obtained the correct answer. A minority of candidates did not make any meaningful attempt in this part, presumably unsure what they needed to do.

Question 5

(a) Most candidates differentiated the given cumulative distribution function to find the probability density function, and then gave a good sketch of this. An acceptable sketch required two straight line sections and labelling to indicate significant points on the axes. Some candidates differentiated correctly but did not attempt the sketch. Other candidates sketched the given cumulative distribution function.

(b) Most candidates answered this part correctly. A minority of candidates integrated the cumulative distribution function instead of the probability density function.

(c) The two quartiles were almost always found correctly, and most candidates subtracted to find the exact value of the interquartile range. Some candidates worked with decimals rather than exact values and were not awarded the final accuracy mark.

Question 6

(a) This part was answered well. Most candidates found the separate unbiased estimators for variance correctly, but these were not always combined correctly for use in the required confidence interval. The common error was to use a pooled estimate. This was not valid because the question stated clearly that it could not be assumed that the population variances were equal. The correct z-value was usually seen in the confidence interval formula. A minority of candidates seemed to be attempting a hypothesis test.

(b) Many candidates found this part more demanding. The majority of candidates standardised appropriately using their value for the standard deviation from **part (a)**. This standardised value was then compared with the critical value 1.282 leading to the conclusion to reject the null hypothesis.

As in earlier questions, there was not always a degree of uncertainty in the conclusion. Often the conclusion was stated in terms of μ_x and μ_y, and not in the context of the question. Some candidates referred to the 'difference' but did not make it clear whether this was between *X* and *Y* or between *Y* and *X*.