FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics Paper 1

Key messages

Candidates should read each question carefully so that they use all the information given and answer all aspects in adequate depth.

Candidates should make clear the method being used and show all the steps in their solutions, particularly when proving a given result.

All sketch graphs should be fully labelled and carefully drawn to show significant points and behaviour at limits.

Both algebra and arithmetic can often be simplified by the use of common factors and the correct use of brackets.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. It seemed that almost all were able to complete the paper in the time allowed. At the higher end, certain questions enabled candidates to demonstrate some elegant approaches.

Comments on specific questions

Question 1

(a) Good responses clearly identified the transformation represented by each matrix before multiplying to show the given answer.

(b) To find a line of invariant points it is necessary to set $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Instead of this, many candidates found an invariant line.

(c) The majority of candidates found the inverse of **M** correctly; a few did not divide by the determinant.

(d) The strongest candidates used the modulus of the determinant when relating the areas, giving $|k| = 3k^2$ and two possible nonzero values of k.

Question 2

Most candidates stated the base case and began the induction by successfully differentiating using the product rule. The inductive hypothesis needed to involve the degree of the polynomial $P_n(x)$. The very strongest responses considered this and clearly justified the $P_n(x)\left(1+x^2\right)^{-n}$ form of the derivative, proving fully the statement given in the question.

Question 3

(a) The majority of candidates substituted $x = y^{\frac{1}{4}}$ to find the required equation and then used the coefficient of y^3 to write down $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. Those who relied solely on formulae for coefficients in terms of the roots were less successful.

(b) The majority of candidates multiplied the original equation by x and rearranged to find the required sum. Only a few incorrectly used the equation derived in **(a)**.

(c) Most successfully applied the formula for the sum of squares of the roots in terms of the coefficients.

Question 4

(a) Most candidates demonstrated the algebraic fluency required and understood that r is a variable and k is a given constant. They could find the partial fractions and use the method of differences to find the sum to n terms.

(b) Those candidates who had a sum in **part (a)** usually found the sum to infinity successfully and used it to find the value of k.

(c) Successful candidates showed a clear method, efficiently subtracting the sum up to $n-1$ from the sum up to n^2. They then used the value of k found in **part (b)** to find the final answer.

Question 5

(a) Almost all substituted $x = r\cos\theta$ and $y = r\sin\theta$ with sufficient working to justify the given polar equation.

(b) Most drew a single loop symmetrical about $\theta = \frac{1}{4}\pi$. The strongest candidates showed the correct behaviour at the pole with the initial line tangential to their loop. Very few candidates gave the answer 3 instead of $\sqrt{3}$ for the greatest distance from the pole.

(c) Almost all chose the correct limits, formed the area and integrated successfully.

(d) Strong responses maintained accuracy when differentiating y with respect to θ. They then used a suitable trigonometry identity and found $\theta = \frac{1}{3}\pi$. This enabled them to find the maximum value of y.

Question 6

(a) Almost all found the correct horizontal and vertical asymptotes.

(b) The differentiation was usually correctly done and the stationary values found.

(c) In most cases the asymptotes were drawn and labelled correctly and the approach to them was acceptable. The branch for $x < \frac{1}{2}$ was the most challenging as it needed to cross the line $y = 2$, have a clearly defined minimum and then approach the line $y = 2$ from below. The single intersection with the axes at $(0, \frac{1}{3})$ was usually found.

(d) This was well answered.

Question 7

There were many very good solutions for this question. Most candidates distinguished correctly between direction vectors and the position vectors of points. A few candidates did not write down the numbers accurately or made arithmetic mistakes when finding the cross-products.

(a) This was usually correct.

(b) The majority of candidates could write down a vector lying in the second plane and use it to find its normal. Then the required angle was usually correct.

(c) Candidates mostly took the correct approach to write down the vector *PQ* in terms of the parameters λ and μ, and they could solve the equations produced by taking the scalar product with the directions of the lines. Some did not finish the question; they needed to find the position vector of either *P* or *Q* as well as the direction of *PQ* to be able to write down the equation.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics Paper 1

Key messages

Candidates should read each question carefully so that they use all the information given and answer all aspects in adequate depth. They should make clear the method being used.

All sketch graphs need to be fully labelled and carefully drawn to show significant points and behaviour at limits.

Candidates should show all the steps in their solutions, particularly when proving a given result.

Both algebra and arithmetic can often be simplified by the use of common factors and brackets, and a simple sketch can help in vector questions.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. It seemed that almost all were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

This was a straightforward induction question and gave the opportunity for candidates to show that they understand the required structure. Successful responses showed the base case for $n = 1$ and stated at the end exactly what has been proved. This involved writing down the result, and not just saying that P_n is true when P_n has not been defined.

Question 2

There were many completely correct solutions for this question

(a) This was performed well with most candidates accurately using the cross product to find the normal and then the equation of the plane. Another successful method was to find three points and use simultaneous equations.

(b) The majority of candidates used the correct process for this question. The most common error was omitting to subtract the angle obtained after using the scalar product. A few used the wrong vectors; candidates who drew a simple sketch to help to understand what each vector represents made these types of mistakes less frequently.

Question 3

(a) Almost all candidates recalled and applied the correct formula.

(b) The simplest solution was to recognise that $\beta + \gamma + \delta = 2 - \alpha$ and so the first three terms can be simplified to $2\alpha^2 - \alpha^3$, for example. Other successful solutions used known formulas. Those candidates who tried to evaluate $\sum \alpha \sum \alpha^2$ often made algebraic mistakes.

(c) (i) The majority found this straightforward. The summation of the equations was done extremely well, the only common problem was to forget that there were 4 roots.

(ii) A correct answer in **(i)** was usually followed by a correct solution here.

Question 4

(a) Most candidates multiplied the first two matrices correctly, with few errors in arithmetic. Successful candidates showed their calculations for the second stage of multiplication before writing down the final answer.

(b) This question was done well. Candidates remembered the difference between invariant points and invariant lines and the majority of solutions were completely correct.

(c) This was well done, with most candidates using the correct terminology and giving the appropriate details of scale factor and direction.

(d) Successful solutions used either an inverse matrix or simultaneous equations. The rare errors were either in the order of the matrices or in forgetting to divide by the determinant when finding the inverse matrix.

Question 5

(a) The majority showed clear use of the method of differences.

(b) Many candidates decided that $|x|<1$ was a necessary condition for the series to converge and gave the sum as 1. Some candidates took note that $x > 0$ was given in the question. Successful solutions recognised that the series also converges if $=1$, with sum 0 in this case.

(c) Successful candidates realised that the laws of logarithms could be used to write $x^{2\log_x r} = r^2$. The first sum to *n* followed. The 'double sum' proved difficult for a significant number who stopped at this point. Some candidates did not fully factorise their answers as instructed in the question.

Question 6

(a) The horizontal asymptote was almost always correct. The most common way of showing that there are no vertical asymptotes was to say that the equation $x^2+1=0$ has no roots.

(b) There were few completely correct answers for this part. A great number of approaches were adopted, with varying degrees of success.

A usually successful solution was to use the rearrangement $y = 1+\dfrac{2}{x^2+1}$ and $0<\dfrac{2}{x^2+1}\leqslant 2$ to prove the result.

Of those who used the discriminant of $(1-y)x^2-y+3=0$, a significant number failed to give proper reasoning to explain why the inequality was strict at one end.

Those candidates who used (0, 3) from **part (c)** did not always give a precise argument as to why this is maximum and how it was connected with the required inequality.

(c) Most differentiated correctly and found the single maximum.

(d) There was enough information from the earlier parts of the question to enable candidates to draw a smooth symmetrical curve with a good approach to the asymptote and a rounded maximum.

(e) The sketch graph was well drawn and many candidates showed correct working to find the critical points and the range of values for x. Those who worked with inequalities were more likely to make errors.

Question 7

(a) Those candidates who started by making substitutions for $\cos\theta$ and $\sin\theta$ took less steps to find a valid cartesian equation. Weaker responses found it difficult to eliminate either r or θ. Once the equation was established most were able to complete the square and therefore give the centre and radius of the circle.

(b) **Part (a)** told the candidates that the curve should be a circle and the majority drew the correct diagram and knew the maximum value of r.

(c) There were many correct solutions to this part. Successful candidates calculated values and explained clearly that the sign of the function changes, in order to justify the existence of a solution.

(d) The majority of candidates integrated the given expressions correctly to find an expression for the area. Most errors were made in finding the limits of integration were the biggest challenge. Those who drew a sketch, or used their diagram from **part (b)** were more frequently successful. In many cases limits of integration were changed several times during the calculation which led to errors. Most candidates realised the connection between the two areas, whose sum is the area of the whole circle, and could write down the second area quickly.

FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics Paper 1

Key messages

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Both algebra and arithmetic can often be simplified by the use of common factors and the correct use of brackets.

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. It seemed that almost all were able to complete the paper in the time allowed. At the higher end, certain questions enabled candidates to demonstrate some elegant approaches.

Comments on specific questions

Question 1

(a) Good responses clearly identified the transformation represented by each matrix before multiplying to show the given answer.

(b) To find a line of invariant points it is necessary to set $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Instead of this, many candidates found an invariant line.

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(d) The strongest candidates used the modulus of the determinant when relating the areas, giving $|k| = 3k^2$ and two possible nonzero values of k.

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Question 3

(a) The majority of candidates substituted $x = y^{\frac{1}{4}}$ to find the required equation and then used the coefficient of y^3 to write down $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. Those who relied solely on formulae for coefficients in terms of the roots were less successful.

(b) The majority of candidates multiplied the original equation by x and rearranged to find the required sum. Only a few incorrectly used the equation derived in **(a)**.

(c) Most successfully applied the formula for the sum of squares of the roots in terms of the coefficients.

Question 4

(a) Most candidates demonstrated the algebraic fluency required and understood that *r* is a variable and *k* is a given constant. They could find the partial fractions and use the method of differences to find the sum to *n* terms.

(b) Those candidates who had a sum in **part (a)** usually found the sum to infinity successfully and used it to find the value of *k*.

(c) Successful candidates showed a clear method, efficiently subtracting the sum up to $n-1$ from the sum up to n^2. They then used the value of *k* found in **part (b)** to find the final answer.

Question 5

(a) Almost all substituted $x = r\cos\theta$ and $y = r\sin\theta$ with sufficient working to justify the given polar equation.

(b) Most drew a single loop symmetrical about $\theta = \frac{1}{4}\pi$. The strongest candidates showed the correct behaviour at the pole with the initial line tangential to their loop. Very few candidates gave the answer 3 instead of $\sqrt{3}$ for the greatest distance from the pole.

(c) Almost all chose the correct limits, formed the area and integrated successfully.

(d) Strong responses maintained accuracy when differentiating y with respect to θ. They then used a suitable trigonometry identity and found $\theta = \frac{1}{3}\pi$. This enabled them to find the maximum value of y.

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There were many very good solutions for this question. Most candidates distinguished correctly between direction vectors and the position vectors of points. A few candidates did not write down the numbers accurately or made arithmetic mistakes when finding the cross-products.

(a) This was usually correct.

(b) The majority of candidates could write down a vector lying in the second plane and use it to find its normal. Then the required angle was usually correct.

(c) Candidates mostly took the correct approach to write down the vector *PQ* in terms of the parameters λ and μ, and they could solve the equations produced by taking the scalar product with the directions of the lines. Some did not finish the question; they needed to find the position vector of either *P* or *Q* as well as the direction of *PQ* to be able to write down the equation.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics Paper 2

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.

Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that all were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers and jumped to conclusions without steps being shown, particularly where answers were given within the question. There were many scripts of a high standard.

Comments on specific questions

Question 1

Almost all candidates found the determinant correctly. While it was common to see row operations or elimination of a variable, only the strongest responses gave a full geometric interpretation by emphasising that the three planes intersect at a unique point.

Question 2

(a) Almost all candidates applied the chain rule to find the first derivative.

(b) Most candidates differentiated the first derivative with respect to t and applied the chain rule again to obtain the correct result.

Question 3

(a) Most candidates accurately recalled the formula for the surface area generated by a curve and showed sufficient working when applying the substitution to justify the given answer.

(b) The majority of candidates used the given substitution and the required hyperbolic identity to simplify the integral to $\int \cosh^2 v \, dv$, which enabled a clear path to the answer. Good responses showed the change of limits clearly using the logarithmic form of $\sinh^{-1} u$.

Question 4

(a) Most candidates multiplied the matrix with the vector to obtain the corresponding eigenvalue.

(b) Most candidates showed enough working when expanding $\det(\mathbf{A} - \lambda\mathbf{I})$ to justify the given characteristic equation. A few did not find the eigenvalues as instructed in the question.

(c) Most candidates were able to maintain accuracy when substituting $\mathbf{A}$ into the characteristic equation, both when multiplying through by $\mathbf{A}^{-1}$ and when making $\mathbf{A}^{-1}$ the subject.

Question 5

Almost all candidates took a correct approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles. Some candidates applied the sum of a geometric series, making clear the first term and common ratio, to accurately derive the given result. Common errors included incorrect areas and numbers of terms.

(b) Most candidates correctly adapted the solution to **(a)** and derived a suitable upper bound. There were some difficulties when simplifying the algebraic expressions and manipulating exponents. Final responses were often in a variety of correct forms.

(c) A minority of candidates showed clearly that the difference between U_N and L_N is proportional to $\frac{1}{N}$, leading efficiently to a lower bound for N.

(d) Few candidates maintained accuracy when manipulating the inequality $L_N < \frac{1}{2\ln 2} < U_N$ and when substituting $N = 500$ into their upper and lower bounds.

Question 7

(a) This was done well with the majority of candidates dividing through by $\sqrt{x^2+16}$ and then arriving at the given integrating factor using the logarithmic form of $\sinh^{-1} x$.

(b) The strongest candidates maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Then the integral of the right hand side could clearly be seen as $\frac{1}{3}x^3 + \frac{1}{3}\left(x^2+16\right)^{\frac{3}{2}} + C$ and the initial conditions substituted to find the correct value of C.

Question 8

(a) After expanding $\left(z+z^{-1}\right)^7$ using the binomial expansion**,** most candidates grouped together terms clearly before applying the identity $z^n + z^{-n} = 2\cos n\theta$ to fully justify their answer.

(b) The majority of successful candidates usually integrated by parts first, using the product $\cos^{n-1}\theta\cos\theta$ to inform their choice of parts, then replaced $\sin^2\theta$ with $1-\cos^2\theta$ to successfully derive the given reduction formula. The few candidates who applied $\cos^2\theta = 1-\sin^2\theta$ first, before integrating by parts, were usually successful also.

(c) Most candidates combined their results from **(a)** and **(b)** to find I_9. Strong responses maintained accuracy throughout, both when integrating to determine I_7 and when substituting into the reduction formula, collecting like terms and giving the final answer in terms of $\sqrt{2}$.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics Paper 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.
- Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. Candidates also showed a thorough understanding of hyperbolic functions and de Moivre's theorem for complex numbers. It seemed that all were able to complete the paper in the time allowed. Most candidates had been well prepared for the examination, but some relied on their calculators too much and did not show sufficient method. There were many scripts of a high standard.

Comments on specific questions

Question 1

Very few incorrect solutions were seen. Some candidates chose to integrate $\frac{1}{\sqrt{(x-5)^2-1}}$ by substitution with a range of substitutions, rather than referring to the List of formulae (MF19).

Question 2

(a) The majority of candidates were able to show the given result convincingly. Some chose to write $\ln xy$ as $\ln x + \ln y$ which made for easier differentiation.

(b) Some candidates chose to rearrange their work to obtain an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. This meant that the quotient formula had to be used as well as implicit differentiation. A more efficient method was to differentiate $8y\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{4}{x}+\frac{4}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=0$ with respect to x.

Question 3

The majority of candidates accurately recalled the formula for the length of the curve with correct limits. Most candidates fully simplified $\sqrt{\dot{x}^2+\dot{y}^2}$ before substituting into the formula, which caused fewer errors and enabled a clear path to the answer.

Question 4

(a) After expanding $(\cos\theta + i\sin\theta)^6$ using the binomial expansion, most candidates accurately grouped together terms contributing to the real and imaginary parts. Good responses then showed clear steps when dividing by $\sin^6\theta$ to fully justify the given result.

(b) Almost all candidates rearranged the polynomial equation correctly, linking with the previous part of the question by substituting $x = \cot\theta$. When solving $\cot 6\theta = 1$, most candidates obtained $x = \cot\left(\frac{1}{24}\pi\right)$. A few candidates gave the values for θ rather than actual solutions as instructed in the question.

Question 5

Almost all candidates took the correct approach to this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. Some candidates omitted the derivative of their general solution. A few candidates gave expressions instead of equations as their answer.

Question 6

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles. Strong responses applied the sum of a geometric series, making clear the first term and common ratio, to accurately derive the given result. Common errors included incorrect areas and numbers of terms.

(b) Most candidates correctly adapted their solution to **(a)** and derived a suitable lower bound. There were some difficulties when simplifying the algebraic expressions which involved exponentials. Final responses were often in a variety of correct forms.

(c) A minority of candidates showed clearly that the difference between U_n and L_n is proportional to $\frac{1}{n}$, hence justifying the given limit.

(d) Most candidates were able to begin the use of Maclaurin's series, but many gave insufficient terms fully justify their answer. Most candidates were able to deduce the required limit and obtain the correct final answer.

Question 7

(a) Almost all candidates were able to produce a correct solution. A few did not show their use of $\sinh 2x = 2\sinh x\cosh x$ to get to the given result.

(b) The majority of candidates divided through by $\sinh 2x$ and then arrived at the correct integrating factor. Successful candidates made the link between this and **part (a)**. Most candidates maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. A few candidates were not able to express the left hand side of the equation as a derivative of the product of the integrating factor and the function. Occasionally the logarithm was incorrectly removed when evaluating the constant.

Question 8

(a) Almost all candidates obtained the characteristic equation in some form. Some candidates did not show sufficient detail in obtaining the given result and some omitted finding the eigenvalues.

(b) Most candidates were able to maintain accuracy throughout their solution, both when substituting into the characteristic equation and when making $\mathbf{A}^3$ the subject before multiplying both sides by $\mathbf{A}$. A few candidates tried multiplying the characteristic equation by λ^{-2}, with little success.

(c) Candidates who used the vector product method to find the eigenvectors tended to be most successful, although sign errors were common. Some evidence was seen of candidates checking that their proposed eigenvector did have the required property by performing matrix multiplication, which allowed some to identify errors. Almost all showed an awareness of how to find the matrices $\mathbf{P}$ and $\mathbf{D}$. A few did not perform the full number of operations on the eigenvalues of $\mathbf{A}$ to form $\mathbf{D}$.

FURTHER MATHEMATICS

Paper 9231/23
Further Pure Mathematics Paper 2

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth, particularly when approximating the area under a curve using rectangles.

Candidates should make use of results derived or given in earlier parts of a question.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that all were able to complete the paper in the time allowed. Sometimes candidates did not fully justify their answers and jumped to conclusions without steps being shown, particularly where answers were given within the question. There were many scripts of a high standard.

Comments on specific questions

Question 1

Almost all candidates found the determinant correctly. While it was common to see row operations or elimination of a variable, only the strongest responses gave a full geometric interpretation by emphasising that the three planes intersect at a unique point.

Question 2

(a) Almost all candidates applied the chain rule to find the first derivative.

(b) Most candidates differentiated the first derivative with respect to t and applied the chain rule again to obtain the correct result.

Question 3

(a) Most candidates accurately recalled the formula for the surface area generated by a curve and showed sufficient working when applying the substitution to justify the given answer.

(b) The majority of candidates used the given substitution and the required hyperbolic identity to simplify the integral to $\int \cosh^2 v \, dv$, which enabled a clear path to the answer. Good responses showed the change of limits clearly using the logarithmic form of $\sinh^{-1} u$.

Question 4

(a) Most candidates multiplied the matrix with the vector to obtain the corresponding eigenvalue.

(b) Most candidates showed enough working when expanding $\det(\mathbf{A} - \lambda\mathbf{I})$ to justify the given characteristic equation. A few did not find the eigenvalues as instructed in the question.

(c) Most candidates were able to maintain accuracy when substituting $\mathbf{A}$ into the characteristic equation, both when multiplying through by $\mathbf{A}^{-1}$ and when making $\mathbf{A}^{-1}$ the subject.

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Almost all candidates took a correct approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the particular integral and some errors when substituting initial conditions. A few candidates gave expressions instead of equations as their answer.

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(b) Most candidates correctly adapted the solution to **(a)** and derived a suitable upper bound. There were some difficulties when simplifying the algebraic expressions and manipulating exponents. Final responses were often in a variety of correct forms.

(c) A minority of candidates showed clearly that the difference between U_N and L_N is proportional to $\frac{1}{N}$, leading efficiently to a lower bound for N.

(d) Few candidates maintained accuracy when manipulating the inequality $L_N < \frac{1}{2\ln 2} < U_N$ and when substituting $N = 500$ into their upper and lower bounds.

Question 7

(a) This was done well with the majority of candidates dividing through by $\sqrt{x^2+16}$ and then arriving at the given integrating factor using the logarithmic form of $\sinh^{-1} x$.

(b) The strongest candidates maintained accuracy and fully simplified the right hand side of the equation after multiplying by the integrating factor. Then the integral of the right hand side could clearly be seen as $\frac{1}{3}x^3 + \frac{1}{3}\left(x^2+16\right)^{\frac{3}{2}} + C$ and the initial conditions substituted to find the correct value of C.

Question 8

(a) After expanding $\left(z+z^{-1}\right)^7$ using the binomial expansion**,** most candidates grouped together terms clearly before applying the identity $z^n + z^{-n} = 2\cos n\theta$ to fully justify their answer.

(b) The majority of successful candidates usually integrated by parts first, using the product $\cos^{n-1}\theta\cos\theta$ to inform their choice of parts, then replaced $\sin^2\theta$ with $1-\cos^2\theta$ to successfully derive the given reduction formula. The few candidates who applied $\cos^2\theta = 1-\sin^2\theta$ first, before integrating by parts, were usually successful also.

(c) Most candidates combined their results from **(a)** and **(b)** to find I_9. Strong responses maintained accuracy throughout, both when integrating to determine I_7 and when substituting into the reduction formula, collecting like terms and giving the final answer in terms of $\sqrt{2}$.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

Candidates are encouraged to pay particular attention in checking that the equations they write are dimensionally correct and consistent.

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates should give sufficient detail in their working so that the offered solution is clear and complete.

General comments

Candidates are encouraged to draw a suitable diagram or, in case a diagram is provided, to annotate it, as this helps understand the problem and model it correctly. For example, in **Question 7(a)** most of the candidates who annotated the diagram identified the components of the velocity of the particle after striking the inclined plane and obtained the correct given answer.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's second law, e.g., in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of well-known formulae.

Comments on specific questions

Question 1

To answer this question the candidates had to use the equation of the trajectory of a projectile $y = x\tan\theta - \frac{gx^2}{2u^2}(1 + \tan^2\theta)$ at points $(56, H)$ and $\left(84, \frac{H}{2}\right)$ and then solve the system of two equations in u and H. Most candidates did this correctly, showing good algebraic manipulative skills. Weaker candidates attempted to obtain the trajectory equation from the formula $s = ut + \frac{1}{2}at$, with limited success.

Question 2

This question required the candidates to write the equation of conservation of mechanical energy at points A and B, to rearrange it to obtain an expression for the velocity at point B, and to substitute it into the equation obtained by applying Newton's second law at point B, to find the tension on the string at point B. In the equation for Newton's second law, a good proportion of candidates used the complementary angle $(90 - \theta)$, or the angle $(90 + \theta)$ however, in these cases, few responses concluded with the correct equation for T.

Some candidates calculated the tension on the string at point A, even though this was not needed to answer the question.

Question 3

(a) To answer this part question the candidates had to perform two steps. First use Hooke's law to work out the extension of the elastic string in equilibrium. Then write and solve the equation of conservation of mechanical energy from the point where the particle is released, until point O. Some candidates did both steps correctly, and obtained the correct answer, often in an elegant way. Most candidates did not perform the first step, and even when the second step was fully correct, did not obtain the correct answer. Some candidates included kinetic energy terms in the equation of conservation of mechanical energy which was not a valid approach for this part.

(b) Most candidates realised that to answer this part question they had to write the equation of conservation of mechanical energy, this time including also the kinetic energy.

Many candidates equated the energy of the system at the point from where the particle was released (d metres below the equilibrium position) to that at the point 2 metres below O, and obtained an equation with gravitational potential energy, elastic potential energy and kinetic energy.

Some candidates realised that if they equated the energy 2 metres below point O to the energy at point O, the equation became much simpler, as it contained only one term for the kinetic energy and one for the gravitational potential energy. They all invariably gained maximum marks.

Question 4

(a) Most candidates set up a dimensionally correct moments equation and many of them obtained the correct answer.

The most common error was to add the moments for the large and small cylinders instead of subtracting them, or to calculate the volume of the object by adding the volumes of the cylinders, instead of subtracting the smaller from the larger.

(b) For this part question many candidates gained at least one mark for setting up the correct equation. Some of them rearranged the equation and simplified it into a quadratic, thus gaining the second mark.

Question 5

(a) In answering this question, many candidates set up the differential equation correctly, however, some did not manage to rearrange the expression in t into a form that would allow it to be integrated. Some candidates integrated the differential equation using the modulus on all natural integrals and invariably gained full marks.

(b) In this part question the candidates used two approaches to obtain an expression for the acceleration. Some candidates substituted their expression for x into the formula provided in **part (a)**, rearranged and simplified it to make v the subject, and differentiated v. Some candidates differentiated their expression for x twice, often incorrectly.

The strongest candidates realised that the magnitude of F cannot be negative and applied the modulus to obtain a positive value.

Question 6

This question proved particularly challenging for candidates. Many of them used the equations for Newton's second law in **part (a)**, the equilibrium of forces in **part (b)**. Not all candidates attempted **part (c)**.

(a) Few candidates realised that the equilibrium of vertical forces on particle P included the effect of the particle Q, through the tension in the string PQ ($T_{OP} \cos\alpha = T_{PQ} \cos\beta + 0.05g$), or by considering the whole system ($T_{OP} \cos\alpha = (0.05 + 0.04)\,g$). These invariably gained full marks.

A typical error was not to take account of this, write $T\cos\alpha = 0.05g$, and then work out $T = \dfrac{0.05g}{\cos\alpha}$ as their answer.

(b) The best responses first applied Newton's second law horizontally, both at P and Q, to eliminate the tension at PQ. The next step was typically to substitute the value of the tension at OP from **part (a)**, and finally to rearrange the equation to obtain the value of ω.

(c) Most of the candidates who made good progress in **parts (a)** and **(b)**, made good progress in this part question too. The best responses divided $T_{PQ}\sin\beta = 0.04 \times 1.4\omega^2$ from **part (b)** by $T_{PQ}\cos\beta = 0.04g$ from part **(a)** to obtain $\tan\beta = \dfrac{7}{4}$ and went on to obtain a correct final answer.

Question 7

(a) This part question proved particularly challenging for some candidates, especially those who did not use a diagram. A variety of equivalent approaches were seen, mostly based on expressing the relationship between the components of the velocity of P parallel and perpendicular to the inclined plane before and after the collision. A less common approach was to use the fact that the velocity vectors of P before and after the collision are perpendicular.

Candidates are always encouraged to show their working clearly. This is particularly important when the answer is given, as in this part question.

(b) Many responses to this part approached the problem using two steps. The first was to determine the height H of the particle just before the impact. Many candidates managed this successfully, using a variety of approaches. The stronger responses then used the answer in the formula $0 = U^2 - 2g \times \dfrac{3}{16}H$, where U is the speed of the particle after the collision, and simplified the resulting equation using the answer from **part (a)**. A common error was to disregard one component of U.

The strongest candidates used an alternative, very elegant, approach to answer the question. They calculated the component of U perpendicular to the plane $\left(U\cos\alpha = \dfrac{3}{5}eu\sin\alpha\right)$, worked out U^2 $\left(U^2 = \dfrac{9}{25}e^2u^2\tan^2\alpha = \dfrac{9}{25}eu^2\right)$ and substituted it into the equation $0 = U^2 - 2g \times \dfrac{3}{16}$, thus obtaining a linear equation in e, which they solved easily.

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics 32

Key messages

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. The number of cases where candidates gave insufficient working to support their answers, with steps in the working missing, was noticeable this session.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is especially the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

In most questions, the majority of candidates understood what method to use. Some omitted to draw a suitable diagram, or to annotate the given diagram, and this resulted in writing incorrect equations.

Candidates are reminded that, when the answer is given, as in **Question 5(a)** they are expected to show their working in full, even if it involves the use of simple algebra.

Comments on specific questions

Question 1

Most candidates made a good attempt at this question, by writing down and equating two expressions for the tension in the string. The first is an application of Hooke's law, the second is an application of Newton's second law of motion.

Question 2

Most candidates realised that they needed to form an energy equation involving elastic potential energy and gravitational potential energy. There is a loss in elastic potential energy from the initial position of the particle to the position where it first comes to instantaneous rest. There is a corresponding gain in gravitational potential energy. A significant number of candidates did not include the initial elastic potential energy. Some candidates included a kinetic energy term in their energy equation, but as the particle is moving from rest to rest, this is incorrect.

Question 3

(a) As always in questions on this topic, equations resulting from the conservation of linear momentum and Newton's law of restitution are required. Along the line of centres, the initial momentum is a difference between two terms and the final momentum is a single term, since B moves perpendicular to the line of centres after the collision in this scenario. In the application of Newton's law of restitution, the initial velocities must be added. The two equations are solved to give a value for $\tan\theta$. The majority of candidates obtained the correct value of $\frac{4}{3}$ for $\tan\theta$. The remaining candidates either had a sign error in one of their equations, or an algebraic error in their working.

(b) Most candidates knew how to work out the initial and final kinetic energies for the spheres, but solutions were often spoiled by algebraic slips.

(c) Most candidates were able to find a value for the angle between the line of centres and A's direction of motion after the collision, but only a minority knew how to find the angle of deflection.

Question 4

(a) The majority of candidates made a reasonable attempt at answering this part. Almost all these candidates were able to resolve vertically and horizontally for the system and obtain two correct equations connecting the forces. These candidates also knew that they needed a moments equation. Most candidates took moments about the point A, so that the forces at A were not involved. However, many had difficulty in finding the moment of the tension, often omitting either the vertical or the horizontal component of the tension.

Some candidates took moments about C and this approach was usually more successful in including all the relevant terms. Having written down three equations (two resolution and one moments), most of these candidates were able to find a value for k.

(b) This part was usually answered well by candidates.

Question 5

(a) Almost all candidates knew how to show the given result. The intended starting point was the equation of the trajectory, given on the formula sheet MF19. Some candidates opted to derive the equation of the trajectory which was not required. A minority of candidates used the equation of the trajectory, but did not show sufficient working to earn full marks. When a result is given, all the essential steps in the working must be shown.

(b) Most candidates attempted this part and scored at least one mark, with many scoring all six marks. Some very concise and well-presented solutions were seen.

The most common method seen was resolving horizontally and vertically to find the relationship between $u\cos\theta$ and $v\cos\alpha$ and the relationship between $u\sin\theta$ and $v\sin\alpha$. These two equations are then squared and added to give an expression for v^2 in terms of a and g. An alternative method is to find the time of flight and the range of the flight for P and then use the fact that the time of flight of Q is half of that for P, while the range is the same for P and Q. A common error in this method was to either use the times of flight for P and Q as equal, or to apply the half to Q instead of P.

Some candidates attempted to use the equation of the trajectory, but this was usually unsuccessful, because the relationship between the times of flight was not used.

Question 6

(a) Most candidates wrote down two energy equations, one for the motion from the initial position to the lowest point and one for the motion from the lowest point to the point where the string goes slack. A third equation, an application of Newton's second law of motion at the point where the string goes slack was also required. From these three equations, a value for $\cos\theta$ can be found. Many candidates applied the method accurately and obtained the correct answer $\cos\theta = \frac{2}{3}$. Some candidates knew the method but made algebraic errors in their solutions.

Some candidates opted to use one energy equation for the motion from the initial position to the point where the string goes slack. This is perfectly valid, but it did lead to errors in calculating the distances involved when considering the change in gravitational potential energy.

(b) A minority candidates found the ratio of the tensions successfully. A common error was to include a component of the tension, usually $T\cos\theta$, instead of simply T, as required at the lowest point of the trajectory.

Question 7

(a) Most candidates were able to set up the correct differential equation, to solve it, including the application of the initial condition, and to find the required expression for v. Sometimes there were sign errors or algebraic errors, but most candidates obtained a logarithmic term on integration. A common error was the omission of a modulus sign in the logarithmic term. Other candidates included the modulus sign but then removed it incorrectly when rearranging to find an expression for v in terms of t.

Candidates are advised to take note of the fact that the result of the integration of a function of the type in this part is given on the formula sheet MF19. There is no need to derive the result using partial fractions.

A few candidates attempted to solve the problem by using *suvat* equations, even though the acceleration was not constant.

(b) Most candidates were again able to set up the correct differential equation, to solve it, including the application of the initial condition, and to find the required expression for v^2 in terms of x. The modulus sign required in the logarithm obtained on integration was often omitted.

FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagram as well.

Candidates are encouraged to pay particular attention in checking that the equations they write are dimensionally correct and consistent.

In all questions, candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates should give sufficient detail in their working so that the offered solution is clear and complete.

General comments

Candidates are encouraged to draw a suitable diagram or, in case a diagram is provided, to annotate it, as this helps understand the problem and model it correctly. For example, in **Question 7(a)** most of the candidates who annotated the diagram identified the components of the velocity of the particle after striking the inclined plane and obtained the correct given answer.

Candidates should be encouraged to check that the equations they write are dimensionally consistent. This is particularly important when writing moments and conservation of energy equations. When applying Newton's second law, e.g., in questions involving collisions, they must ensure they explicitly mention the mass or masses involved.

Candidates should be reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of well-known formulae.

Comments on specific questions

Question 1

To answer this question the candidates had to use the equation of the trajectory of a projectile $y = x\tan\theta - \dfrac{gx^2}{2u^2}(1 + \tan^2\theta)$ at points $(56, H)$ and $\left(84, \dfrac{H}{2}\right)$ and then solve the system of two equations in u and H. Most candidates did this correctly, showing good algebraic manipulative skills. Weaker candidates attempted to obtain the trajectory equation from the formula $s = ut + \dfrac{1}{2}at$, with limited success.

Question 2

This question required the candidates to write the equation of conservation of mechanical energy at points A and B, to rearrange it to obtain an expression for the velocity at point B, and to substitute it into the equation obtained by applying Newton's second law at point B, to find the tension on the string at point B. In the equation for Newton's second law, a good proportion of candidates used the complementary angle $(90 - \theta)$, or the angle $(90 + \theta)$ however, in these cases, few responses concluded with the correct equation for T.

Some candidates calculated the tension on the string at point A, even though this was not needed to answer the question.

Question 3

(a) To answer this part question the candidates had to perform two steps. First use Hooke's law to work out the extension of the elastic string in equilibrium. Then write and solve the equation of conservation of mechanical energy from the point where the particle is released, until point O. Some candidates did both steps correctly, and obtained the correct answer, often in an elegant way. Most candidates did not perform the first step, and even when the second step was fully correct, did not obtain the correct answer. Some candidates included kinetic energy terms in the equation of conservation of mechanical energy which was not a valid approach for this part.

(b) Most candidates realised that to answer this part question they had to write the equation of conservation of mechanical energy, this time including also the kinetic energy.

Many candidates equated the energy of the system at the point from where the particle was released (d metres below the equilibrium position) to that at the point 2 metres below O, and obtained an equation with gravitational potential energy, elastic potential energy and kinetic energy.

Some candidates realised that if they equated the energy 2 metres below point O to the energy at point O, the equation became much simpler, as it contained only one term for the kinetic energy and one for the gravitational potential energy. They all invariably gained maximum marks.

Question 4

(a) Most candidates set up a dimensionally correct moments equation and many of them obtained the correct answer.

The most common error was to add the moments for the large and small cylinders instead of subtracting them, or to calculate the volume of the object by adding the volumes of the cylinders, instead of subtracting the smaller from the larger.

(b) For this part question many candidates gained at least one mark for setting up the correct equation. Some of them rearranged the equation and simplified it into a quadratic, thus gaining the second mark.

Question 5

(a) In answering this question, many candidates set up the differential equation correctly, however, some did not manage to rearrange the expression in t into a form that would allow it to be integrated. Some candidates integrated the differential equation using the modulus on all natural integrals and invariably gained full marks.

(b) In this part question the candidates used two approaches to obtain an expression for the acceleration. Some candidates substituted their expression for x into the formula provided in **part (a)**, rearranged and simplified it to make v the subject, and differentiated v. Some candidates differentiated their expression for x twice, often incorrectly.

The strongest candidates realised that the magnitude of F cannot be negative and applied the modulus to obtain a positive value.

Question 6

This question proved particularly challenging for candidates. Many of them used the equations for Newton's second law in **part (a)**, the equilibrium of forces in **part (b)**. Not all candidates attempted **part (c)**.

(a) Few candidates realised that the equilibrium of vertical forces on particle P included the effect of the particle Q, through the tension in the string PQ ($T_{OP}\cos\alpha = T_{PQ}\cos\beta + 0.05g$), or by considering the whole system ($T_{OP}\cos\alpha = (0.05 + 0.04)g$). These invariably gained full marks.

A typical error was not to take account of this, write $T\cos\alpha = 0.05g$, and then work out $T = \dfrac{0.05g}{\cos\alpha}$ as their answer.

(b) The best responses first applied Newton's second law horizontally, both at P and Q, to eliminate the tension at PQ. The next step was typically to substitute the value of the tension at OP from **part (a)**, and finally to rearrange the equation to obtain the value of ω.

(c) Most of the candidates who made good progress in **parts (a)** and **(b)**, made good progress in this part question too. The best responses divided $T_{PQ}\sin\beta = 0.04 \times 1.4\omega^2$ from **part (b)** by $T_{PQ}\cos\beta = 0.04g$ from part **(a)** to obtain $\tan\beta = \dfrac{7}{4}$ and went on to obtain a correct final answer.

Question 7

(a) This part question proved particularly challenging for some candidates, especially those who did not use a diagram. A variety of equivalent approaches were seen, mostly based on expressing the relationship between the components of the velocity of P parallel and perpendicular to the inclined plane before and after the collision. A less common approach was to use the fact that the velocity vectors of P before and after the collision are perpendicular.

Candidates are always encouraged to show their working clearly. This is particularly important when the answer is given, as in this part question.

(b) Many responses to this part approached the problem using two steps. The first was to determine the height H of the particle just before the impact. Many candidates managed this successfully, using a variety of approaches. The stronger responses then used the answer in the formula $0 = U^2 - 2g \times \dfrac{3}{16}H$, where U is the speed of the particle after the collision, and simplified the resulting equation using the answer from **part (a)**. A common error was to disregard one component of U.

The strongest candidates used an alternative, very elegant, approach to answer the question. They calculated the component of U perpendicular to the plane $\left(U\cos\alpha = \dfrac{3}{5}eu\sin\alpha\right)$, worked out U^2 $\left(U^2 = \dfrac{9}{25}e^2u^2\tan^2\alpha = \dfrac{9}{25}eu^2\right)$ and substituted it into the equation $0 = U^2 - 2g \times \dfrac{3}{16}$, thus obtaining a linear equation in e, which they solved easily.

FURTHER MATHEMATICS

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Further Probability & Statistics 41

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates should give sufficient detail in their working so that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems, where several different values are calculated, each depending on the previous one. Such rounding errors were sometimes seen in **Questions 3(b)**.

Comments on specific questions

Question 1

Most candidates answered this question well. Some candidates used an incorrect *z*-value in the formula for the confidence interval and a few candidates made arithmetical errors in calculating the sample variance from the given data.

Question 2

(a) Most candidates knew the method for carrying out a Wilcoxon matched-pairs signed-rank test. They found the signed differences, ranked them appropriately and identified the correct test statistic as 20. They then compared this with the critical value of 17 from tables, and stated the conclusion; to accept the null hypothesis.

The mark for the hypotheses was not awarded very often. Candidates need to be more aware that hypotheses for parametric tests must be stated in terms of the *population* median, not simply the median. Some candidates used the incorrect parameter μ in the hypotheses, which is the standard notation for population mean not population median.

Conclusions were not always given with any degree of uncertainty, with words such as 'prove' being used.

(b) Most candidates deduced that the error in the recording of the mark for candidate C would only affect the signed rank of that entry and not its position in the ranking. This would result in a test statistic of 13 instead of 20. The critical value is unchanged at 17, so the conclusion of the test would now be reversed as 13 is less than 17, whereas 20 is greater than 17.

Question 3

(a) Answers to this part were almost always correct. There were a few arithmetical slips seen.

(b) Most candidates scored full marks in this part. They stated appropriate hypotheses for the test and used the given information to find the chi-squared test statistic accurately. Some candidates made errors in the use of the formula for the chi-squared value, for example by having the observed values in the denominator. Other candidates did not work to sufficient accuracy.

The hypotheses were sometimes incomplete with the null hypothesis being 'It is a good fit'. A complete statement such as 'Po(1.9) is a good fit for the data' is required, and similarly for the alternative hypothesis.

Usually, the test statistic was compared with the correct critical value of 7.779 and a correct decision made about H_0. Many candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language.

Question 4

(a) Only a minority of candidates produced a correct sketch in this part. Many recognised that the function was a curve from $x = 0$ to $x = 1$ and a straight line segment from $x = 1$ to $x = 5$, but most joined these at $x = 1$. This is incorrect as the line segment has initial coordinates $(1, 4k)$ while the curve has coordinates $(1, k)$.

(b) Most candidates were able to find the value of k by using the fact that the area under the curve of the probability density function is equal to one.

(c) Almost all candidates integrated the probability density function, and did so correctly, but a common error was to omit the constant term. It was also necessary to include $\mathrm{F}(x) = 0$ for $x < 0$ and $\mathrm{F}(x) = 1$ for $x > 5$, but often these were incorrectly combined as '$\mathrm{F}(x) = 0$ otherwise' or omitted altogether.

(d) Most candidates knew how to proceed in this part. Some candidates realised that the median must lie in the second part of the piecewise function and equated their cumulative distribution function to 0.5 and solved the resulting equation. Other candidates worked with the probability density function. A few candidates equated the probability density function, rather than the cumulative distribution function, to 0.5.

Question 5

(a) This part was answered well by the majority of candidates.

(b) Most candidates wrote down the probability generating function for Y correctly and then found the probability generating function for Z by multiplying those for X and Y. Any errors were usually arithmetical or algebraic.

(c) Most candidates differentiated their result from **part (b)** and substituted in $t = 1$ to find the value of $\mathrm{E}(Z)$.

Question 6

Most candidates offered good solutions to this question, with the majority gaining full marks. Most candidates found the two individual sample variances explicitly, then a pooled variance for the combined distribution. The question states clearly that the two distributions have equal population variances, so a pooled variance should be used. Some candidates used a two-sample non-pooled variance despite the question stating that the two distributions have equal population variances.

Some candidates chose to leave any explicit calculations until the final step of finding the test statistic. Candidates are reminded the need to clearly communicate their understanding by showing detail in their responses. If method is clear in candidates' working, credit can be awarded for understanding even if the final answer is not correct.

Candidates are advised to evaluate quantities as they progress through the method, retaining a good degree of accuracy as they do so. This accuracy needs to be at least 4 significant figures so that their final answer is accurate to the required 3 significant figures.

Candidates completed the test by finding the test statistic and comparing it with the correct critical value of 1.341. The final mark for giving the conclusion of the test in context could not always be awarded, even though all the working was correct, as some candidates gave conclusions that lacked any uncertainty in the language used, for example by using the word 'prove'.

FURTHER MATHEMATICS

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Further Probability & Statistics 42

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates should give sufficient detail in their working so that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that….' rather than 'the test proves that….'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems, where several different values are calculated, each depending on the previous one. Such rounding errors were sometimes seen in **Questions 3(b) and 5**.

Comments on specific questions

Question 1

Most candidates answered this question well. Some candidates used a z-value instead of a t-value in the formula for the confidence interval and a few candidates made arithmetical errors in calculating the sample variance from the given data.

Question 2

(a) Most candidates answered this well. A minority of candidates were able to deduce that $p+2q=1.1$ but were not able to find the second relationship between p and q which comes from the sum of the coefficients in the probability generating function summing to one.

(b) The majority of candidates knew what was required in this part. Some candidates substituted the values of p and q found in **part (a)** the wrong way round, and others lost the t outside the given expression for the probability generating function for Y.

(c) Candidates knew that they needed to sum the coefficients of t^3, t^4 and t^5 in their expression in **part (b)**. Errors were either arithmetical or due to the inclusion of the coefficient of t^2.

(d) This part was answered well by the majority of candidates.

Question 3

(a) Many candidates correctly calculated the mean value from the data given in the table, and then used the fact that the mean value is equal to np, where n is the total number of pots and p is the probability used in the Binomial distribution $B(n, p)$. As the answer is given, working needed to be seen for the mark to be awarded. A minority of candidates did not attempt this part, and others made arithmetical errors.

(b) Most candidates stated appropriate hypotheses for the test, calculated and showed the correct expected frequencies, with most going on to find the test statistic accurately. A small number made errors in the use of the formula, for example by having the observed values in the denominator. Other candidates did not work to sufficient accuracy, sometimes using only one decimal place in their intermediate calculations leading to an inaccurate test statistic. Usually, the test statistic was compared with the correct critical value and a correct decision made about H_0. Many candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language. An example of an incorrect statement is that there is '*sufficient* evidence to suggest that B(5, 0.42) is a good fit to the data' rather than the correct statement '*insufficient* evidence to suggest that B(5, 0.42) is not a good fit to the data'.

Question 4

(a) Almost all candidates integrated the probability density function, and did so correctly, but a common error was to omit the constant term. It was also necessary to include $F(x) = 0$ for $x < 2$ and $F(x) = 1$ for $x > 5$, but often these were incorrectly combined as '$F(x) = 0$ otherwise' or omitted altogether.

Some candidates made the integration more complicated than necessary by expanding the squared bracket first. Although this was not too much of a problem in this part, it made subsequent parts of the question more complex.

(b) The method for this part was known, though as mentioned earlier, for some the algebra was more complex than it needed to be. A minority of candidates correctly found the cumulative distribution function for Y but did not go on to differentiate to find the corresponding probability density function.

(c) Most candidates knew how to proceed in this part, by equating their cumulative distribution function for Y to 0.5 and solving the resulting equation. A few candidates equated the probability density function, rather than the cumulative distribution function, to 0.5.

(d) Most candidates knew the method in this part and many obtained the correct answer.

Question 5

Most candidates made good progress in this question. They found the two individual sample variances explicitly, then a variance appropriate for a two-sample test. Then they found the correct test statistic of –2.27. Some candidates used a pooled variance in their calculation, even though there was no indication in the question that the two population variances could be assumed to be equal.

Some candidates chose to leave any explicit calculations until the final step of finding the test statistic. Candidates are reminded the need to clearly communicate their understanding by showing detail in their responses. If method is clear in candidates' working, credit can be awarded for understanding even if the final answer is not correct.

Candidates are advised to evaluate quantities as they progress through the method, retaining a good degree of accuracy as they do so. This accuracy needs to be at least 4 significant figures so that their final answer is accurate to the required 3 significant figures.

The majority of candidates went on to find the set of possible values of α which would suggest that there is sufficient evidence to support Dev's claim. A common error at this stage was to give $\alpha < 1.15$ rather than the correct $\alpha > 1.15$. The remaining candidates did not seem to know how to approach the last part of the question.

Question 6

Candidates should be aware that hypotheses for parametric tests must be stated in terms of *population* parameters; a significant number of candidates referred to ‘median’ rather than ‘population median’. Others used the incorrect parameter μ which is the standard notation for population mean not population median.

The vast majority of candidates were able to find the signed differences, rank these appropriately from 1 to 22 and identify the correct test statistic of 102. Most candidates knew that they needed to use a normal approximation at this stage, and correctly found the mean and variance of the distribution. Some candidates did not include a continuity correction when calculating the test statistic.

Most candidates compared their test statistic with the correct critical *z*-value and made the appropriate decision to accept H_0. However, the final mark for the conclusion in context could not always be awarded as candidates either used definite language (such as ‘prove’) or did not use uncertainty in their language, such as ‘insufficient evidence’. Some stated that there was ‘sufficient evidence to reject the claim’ instead of ‘insufficient evidence to support the claim’.

FURTHER MATHEMATICS

Paper 9231/43
Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working, as credit is given for method as well as accuracy. When a result is given in a question, candidates should give sufficient detail in their working so that the offered solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, ‘there is insufficient evidence to support the claim that….’ rather than ‘the test proves that….’.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems, where several different values are calculated, each depending on the previous one. Such rounding errors were sometimes seen in **Questions 3(b)**.

Comments on specific questions

Question 1

Most candidates answered this question well. Some candidates used an incorrect *z*-value in the formula for the confidence interval and a few candidates made arithmetical errors in calculating the sample variance from the given data.

Question 2

(a) Most candidates knew the method for carrying out a Wilcoxon matched-pairs signed-rank test. They found the signed differences, ranked them appropriately and identified the correct test statistic as 20. They then compared this with the critical value of 17 from tables, and stated the conclusion; to accept the null hypothesis.

The mark for the hypotheses was not awarded very often. Candidates need to be more aware that hypotheses for parametric tests must be stated in terms of the *population* median, not simply the median. Some candidates used the incorrect parameter μ in the hypotheses, which is the standard notation for population mean not population median.

Conclusions were not always given with any degree of uncertainty, with words such as ‘prove’ being used.

(b) Most candidates deduced that the error in the recording of the mark for candidate *C* would only affect the signed rank of that entry and not its position in the ranking. This would result in a test statistic of 13 instead of 20. The critical value is unchanged at 17, so the conclusion of the test would now be reversed as 13 is less than 17, whereas 20 is greater than 17.

Question 3

(a) Answers to this part were almost always correct. There were a few arithmetical slips seen.

(b) Most candidates scored full marks in this part. They stated appropriate hypotheses for the test and used the given information to find the chi-squared test statistic accurately. Some candidates made errors in the use of the formula for the chi-squared value, for example by having the observed values in the denominator. Other candidates did not work to sufficient accuracy.

The hypotheses were sometimes incomplete with the null hypothesis being 'It is a good fit'. A complete statement such as 'Po(1.9) is a good fit for the data' is required, and similarly for the alternative hypothesis.

Usually, the test statistic was compared with the correct critical value of 7.779 and a correct decision made about H_0. Many candidates attempted to give a conclusion in context, as is required, with an appropriate amount of uncertainty expressed in the language.

Question 4

(a) Only a minority of candidates produced a correct sketch in this part. Many recognised that the function was a curve from $x = 0$ to $x = 1$ and a straight line segment from $x = 1$ to $x = 5$, but most joined these at $x = 1$. This is incorrect as the line segment has initial coordinates $(1, 4k)$ while the curve has coordinates $(1, k)$.

(b) Most candidates were able to find the value of k by using the fact that the area under the curve of the probability density function is equal to one.

(c) Almost all candidates integrated the probability density function, and did so correctly, but a common error was to omit the constant term. It was also necessary to include $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 5$, but often these were incorrectly combined as '$F(x) = 0$ otherwise' or omitted altogether.

(d) Most candidates knew how to proceed in this part. Some candidates realised that the median must lie in the second part of the piecewise function and equated their cumulative distribution function to 0.5 and solved the resulting equation. Other candidates worked with the probability density function. A few candidates equated the probability density function, rather than the cumulative distribution function, to 0.5.

Question 5

(a) This part was answered well by the majority of candidates.

(b) Most candidates wrote down the probability generating function for *Y* correctly and then found the probability generating function for *Z* by multiplying those for *X* and *Y*. Any errors were usually arithmetical or algebraic.

(c) Most candidates differentiated their result from **part (b)** and substituted in $t = 1$ to find the value of $E(Z)$.

Question 6

Most candidates offered good solutions to this question, with the majority gaining full marks. Most candidates found the two individual sample variances explicitly, then a pooled variance for the combined distribution. The question states clearly that the two distributions have equal population variances, so a pooled variance should be used. Some candidates used a two-sample non-pooled variance despite the question stating that the two distributions have equal population variances.

Some candidates chose to leave any explicit calculations until the final step of finding the test statistic. Candidates are reminded the need to clearly communicate their understanding by showing detail in their responses. If method is clear in candidates' working, credit can be awarded for understanding even if the final answer is not correct.

Candidates are advised to evaluate quantities as they progress through the method, retaining a good degree of accuracy as they do so. This accuracy needs to be at least 4 significant figures so that their final answer is accurate to the required 3 significant figures.

Candidates completed the test by finding the test statistic and comparing it with the correct critical value of 1.341. The final mark for giving the conclusion of the test in context could not always be awarded, even though all the working was correct, as some candidates gave conclusions that lacked any uncertainty in the language used, for example by using the word 'prove'.